Towards a directed HoTT based on 4 kinds of variance

Andreas Nuyts

Promoter: Frank Piessens
Supervisors: Jesper Cockx, Dominique Devriese
Dept. Computer Science, KU Leuven

June 29, 2015

KU LEUVEN

Table of Contents

KU LEUVEN

Directed HoTT: What and why?

HoTT	∞-groupoids	Homotopy Theory
type A	∞-groupoid A	space A
element $a : A$	object $a \in \mathbf{obj}(A)$	point $a \in A$
$p : a =_A b$	isomorphism $p : a \to b$	path from a to b
function $A \to B$	functor $A \to B$	continuous map $A \to B$
Directed HoTT	(ω, ω)-categories	Directed Homot. Theory
$\varphi : a \leadsto_A b$	morphism $\varphi : a \to b$	directed path from a to b

Why?

- Math: Constructive theory for (ω, ω)-categories,
- Math: Domain-specific morphism is type-theoretic morphism,
- Progr: Automatic implementation of `fmap` for functions (but: canonicity problem),
- Bonus: Constructive (co)limits.

KU LEUVEN

The approach

- Knowledge from (higher) category theory was not central in the development.
- Made sense of variance for dependent functions,
- Investigated the variance of important dependencies in HoTT,
- Adapted foundations of HoTT so as to incorporate functoriality from the start,
- Generalized univalence to give meaning to paths & morphisms,
- Inductive definition of morphisms with induction principle turns out to be possible.

Work in progress: Still a bit shabby.

- Semantics/consistency?
- Mysterious extra structure on types (zigzags?)

KU LEUVEN

2D Directed Type Theory (Licata & Harper, 2011)

Directed:

- For all $a, b : A$, we can speak of morphisms $a \rightsquigarrow b$.

2-dimensional:

- Distinct elements $a, b : A$,
- Distinct morphisms $\varphi, \chi : a \rightsquigarrow b$,
- Either $\varphi \equiv \chi$ or not.

New: Co/Contravariance of assumptions $(x :^{\pm} A)$:

$$\frac{\Gamma, x :^{-} A \vdash B[x]\,\textbf{type}}{\Gamma \vdash \prod_{x:A} B[x]\,\textbf{type}}$$

Opportunities for improvement:

- 2DTT excludes identity type. E.g. let $a = \square$ be covariant. $\varphi : a \rightsquigarrow b$ gives $\varphi_* : (a = a) \to (a = b)$, so $\varphi_*(\textbf{refl}\, a) : a = b$. $a = \square$ should be **invariant**.
- Decouple variance in types and in elements. 2DTT: $\prod_{a:A} C(a)$ contravariant in A. Therefore: $C(a)$ and $f(a)$. Remarkably: $\prod^{+}$ that is covariant in *domain*.

Meaning of structure: univalence

In HoTT:

- 2 structures: paths and functions.
- Univalence: path in universe is invertible function:
  $(A =_{\mathbf{U}} B) \simeq (A \simeq B)$.

In directed HoTT

- 3 structures: paths, morphisms and covariant functions.
- Directed univalence: morphism in universe is covariant function:
  $(A \rightsquigarrow_{\mathbf{U}} B) \simeq (A \xrightarrow{+} B)$.
- Categorical univalence: path in *any type* is invertible morphism:
  $(a =_A b) \simeq (a \cong_A b)$.
  (Cfr. precategory vs. category)

KU LEUVEN

Variance

At least 4 kinds of variance appear in HoTT:

Covariant:	$f : A \xrightarrow{+} C$	$a \rightsquigarrow_A b$	implies	$f(a) \rightsquigarrow_C f(b)$,
Contravariant:	$f : A \xrightarrow{-} C$	$a \rightsquigarrow_A b$	implies	$f(a) \leftsquigarrow_C f(b)$,
Invariant:	$f : A \xrightarrow{\times} C$	$a \rightsquigarrow_A b$	implies	(almost) nothing,
Isovariant:	$f : A \xrightarrow{=} C$	$a \rightsquigarrow_A b$	implies	$f(a) =_C f(b)$.

All functions preserve equality ($\simeq$ isomorphism), as in HoTT.

Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms.
Let $C : A \xrightarrow{v} \mathbf{U}$.

$$\text{Any } f \text{ yields} \frac{f(a) =_C f(b)}{p : a =_A b},$$
$$\text{Isovariant } f \text{ yields} \frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},$$
$$\text{Covariant } f \text{ yields} \frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},$$
$$\text{Contravariant } f \text{ yields} \frac{f(a) \leftsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.$$

Meaning depends on variance of C.

Whenever $p : a =_A b$,

$$p_* : C(a) \simeq C(b),$$

$$\left(\frac{c =_C d}{p : a =_A b}\right) \simeq \left(p_*(c) =_{C(b)} d\right).$$

Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms.
Let $C : A \xrightarrow{v} \mathbf{U}$.

$$\text{Any } f \text{ yields} \frac{f(a) =_C f(b)}{p : a =_A b},$$

$$\text{Isovariant } f \text{ yields} \frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Covariant } f \text{ yields} \frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Contravariant } f \text{ yields} \frac{f(a) \leftsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.$$

Meaning depends on variance of C.

Assume $C(x)$ covariant in x.
Whenever $\varphi : a \rightsquigarrow_A b$,

$$\varphi_* : C(a) \xrightarrow{+} C(b),$$

$$\left(\frac{c =_C d}{\varphi : a \rightsquigarrow_A b}\right) \simeq \left(\varphi_*(c) =_{C(b)} d\right).$$

KU LEUVEN

Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms.
Let $C : A \xrightarrow{v} \mathbf{U}$.

$$\text{Any } f \text{ yields} \frac{f(a) =_C f(b)}{p : a =_A b},$$

$$\text{Isovariant } f \text{ yields} \frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Covariant } f \text{ yields} \frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Contravariant } f \text{ yields} \frac{f(a) \leftsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.$$

Meaning depends on variance of C.

Assume $C(x)$ contravariant in x.
Whenever $\varphi : a \rightsquigarrow_A b$,

$$\varphi^* : C(b) \xrightarrow{+} C(a),$$

$$\left(\frac{c =_C d}{\varphi : a \rightsquigarrow_A b}\right) \simeq \left(c =_{C(a)} \varphi^*(d)\right).$$

KU LEUVEN

Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms.
Let $C : A \xrightarrow{v} \mathbf{U}$.

$$\text{Any } f \text{ yields} \frac{f(a) =_C f(b)}{p : a =_A b},$$

$$\text{Isovariant } f \text{ yields} \frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Covariant } f \text{ yields} \frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Contravariant } f \text{ yields} \frac{f(a) \leftsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.$$

Meaning depends on variance of C.

Assume $C(x)$ isovariant in x.
Whenever $\varphi : a \rightsquigarrow_A b$,

$$\varphi_* : C(a) \simeq C(b),$$

$$\left(\frac{c =_C d}{\varphi : a \rightsquigarrow_A b}\right) \simeq \left(\varphi_*(c) =_{C(b)} d\right).$$

Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms.
Let $C : A \xrightarrow{v} \mathbf{U}$.

$$\text{Any } f \text{ yields} \frac{f(a) =_C f(b)}{p : a =_A b},$$
$$\text{Isovariant } f \text{ yields} \frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},$$
$$\text{Covariant } f \text{ yields} \frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},$$
$$\text{Contravariant } f \text{ yields} \frac{f(a) \leftsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.$$

Meaning depends on variance of C.

When $C(x)$ invariant in x:

- Heterogeneous types are still defined and often have a meaning.

Invariant type families map morphisms to relations?
Invariant functions map morphisms to

- spans: $a \leftsquigarrow * \rightsquigarrow b$?
- cospans: $a \rightsquigarrow * \leftsquigarrow b$?
- zigzags: $a \rightsquigarrow * \leftsquigarrow \ldots \rightsquigarrow * \leftsquigarrow b$?

KU LEUVEN

Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms.
Let $C : A \xrightarrow{v} \mathbf{U}$.

$$\text{Any } f \text{ yields } \frac{f(a) =_C f(b)}{p : a =_A b},$$

$$\text{Isovariant } f \text{ yields } \frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Covariant } f \text{ yields } \frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},$$

$$\text{Contravariant } f \text{ yields } \frac{f(a) \leftsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.$$

Meaning depends on variance of C.

Example:
$C : (X : \mathbf{U}) \overset{\times}{\mapsto} \left(X \xrightarrow{+} X \right)$.

Given morphism $h : A \xrightarrow{+} B$,

$$\left(A \xrightarrow{+} A \right) \xrightarrow{h\circ} \left(A \xrightarrow{+} B \right) \xleftarrow{\circ h} \left(B \xrightarrow{+} B \right)$$

$$\left(\frac{f =_C g}{\mathbf{dua}(h) : A \rightsquigarrow B} \right) \simeq (h \circ f = g \circ h).$$

Commutative diagram = path along morphism
Commutative diagram for every morphism? **Isovariance.** E.g. $\mathbf{id}_X$.

KU LEUVEN

Example: a cospan type

Idea: $(a \curlywedge b) \simeq \sum_{c:A}(a \rightsquigarrow c) \times (b \rightsquigarrow c)$

Inductive type family

Cospan type family $a \curlywedge_A b$ has constructors

- **triv** : $\prod^{+}_{a:A} a \curlywedge a$,
- Contravariance in a: $(a' \rightsquigarrow a) \xrightarrow{+} (a \curlywedge b) \xrightarrow{+} (a' \curlywedge b)$,
- Contravariance in b: $(b' \rightsquigarrow b) \xrightarrow{+} (a \curlywedge b) \xrightarrow{+} (a \curlywedge b')$.

To define $f : \prod^{?}_{a,b:A} \prod^{+}_{w:a \curlywedge b} \xrightarrow{+} C(a,b,w)$, you need:

- $f_{\mathbf{triv}} : \prod^{+}_{a:A} C(a, a, \mathbf{triv}\, a)$,
- $C(a,b,w)$ contravariant in a and b.

Computation:

- $f(a, a, \mathbf{triv}\, a) \equiv f_{\mathbf{triv}}(a)$,
- f is isovariant in a and b.

KU LEUVEN

The morphism type

Inductive type family

Morphism type family $a \rightsquigarrow_A b$ has constructors

- $\mathbf{id} : \prod^{=}_{a:A} a \rightsquigarrow a$,
- Contravariance in a: $(a' \rightsquigarrow a) \xrightarrow{+} (a \rightsquigarrow b) \xrightarrow{+} (a' \rightsquigarrow b)$,
- Covariance in b: $(b \rightsquigarrow b') \xrightarrow{+} (a \rightsquigarrow b) \xrightarrow{+} (a \rightsquigarrow b')$.

To define $f : \prod^{?}_{a,b:A} \prod^{+}_{\varphi : a \rightsquigarrow b} \xrightarrow{+} C(a, b, \varphi)$, you need:

- $f_{\mathbf{id}} : \prod^{=}_{a:A} C(a, a, \mathbf{id}\, a)$,
- $C(a, b, \varphi)$ contravariant in a and covariant in b.

Computation:

- $f(a, a, \mathbf{id}\, a) \equiv f_{\mathbf{id}}(a)$,
- f is isovariant in a and b.

KU LEUVEN

The identity type

Inductive type family

Identity type family $a =_A b$ has constructors

- **refl** : $\prod^{=}_{a:A} a = a$,
- Invariance in a and b.

Although invariance cannot be written as a constructor, the scheme seems to generalize:

To define $f : \prod^{?}_{a,b:A} \prod^{+}_{p:a=b} \xrightarrow{+} C(a, b, p)$, you need:

- $f_{\mathbf{refl}} : \prod^{=}_{a:A} C(a, a, \mathbf{refl}\, a)$,
- $C(a, b, p)$ invariant in a and b (this condition is void).

Computation:

- $f(a, a, \mathbf{refl}\, a) \equiv f_{\mathbf{refl}}(a)$,
- f is isovariant in a and b.

KU LEUVEN

Isovariance of **refl** and **id**

- **id** a is isovariant in a, because $\dfrac{\mathbf{id}\, a = \mathbf{id}\, b}{\varphi : a \rightsquigarrow b}$ is equivalent to commutativity of:

$$\begin{array}{ccc} a & \overset{\varphi}{\rightsquigarrow} & b \\ {\scriptstyle \mathbf{id}\, a}\,\wr & & \wr\,{\scriptstyle \mathbf{id}\, b} \\ a & \underset{\varphi}{\rightsquigarrow} & b. \end{array}$$

- **refl** a is isovariant in a, because (interestingly!) $\dfrac{\mathbf{refl}\, a = \mathbf{refl}\, b}{\varphi : a \rightsquigarrow b}$ is equivalent to commutativity of:

$$\begin{array}{ccc} a & \overset{\varphi}{\rightsquigarrow} & b \\ {\scriptstyle \mathbf{refl}\, a}\,\Vert & & \Vert\,{\scriptstyle \mathbf{refl}\, b} \\ a & \underset{\varphi}{\rightsquigarrow} & b. \end{array}$$

The mystery of invariance, zigzags and relations

Questions:

- Are zigzags the way to go?
- What are morphisms between zigzags?
- Well-behaved type-theoretic account of zigzags?
  - Must not prove $\zeta^\dagger \circ \zeta = \mathbf{id}$.
- Analogue of opposite for invariance?
  - $(A \xrightarrow{-} B) \stackrel{+}{\simeq} (A^{\mathbf{op}} \xrightarrow{+} B) \stackrel{-}{\simeq} (A \xrightarrow{+} B^{\mathbf{op}})$,
  - $(A \xrightarrow{=} B) \stackrel{+}{\simeq} (A^{\mathbf{loc}} \xrightarrow{+} B) \stackrel{\times}{\simeq} (A \xrightarrow{+} B^{\mathbf{core}})$,
  - $f : (A^{\mathbf{core}} \xrightarrow{+} B)$ is function $A \to B$ that discards morphisms completely.
  - $g : (A \xrightarrow{+} B^{\mathbf{loc}})$ is function $A \to B$ that maps all structure to zigzags.
  - Can we have a core with degenerate/infinitesimal zigzags?
- ...

KU LEUVEN

Other results

- Directed function extensionality: Morphism of functions is a natural transformation,
- $(\prod^{=}_{a:A} C(a)) = (\varprojlim_{a:A} C(a))$ for covariant C,
- $(\sum^{=}_{a:A} C(a)) = (\varinjlim_{a:A} C(a))$ for covariant C,
- Some functions, such as $A \mapsto A^{\mathbf{op}}$, have complicated 'higher' variance.

Thanks!

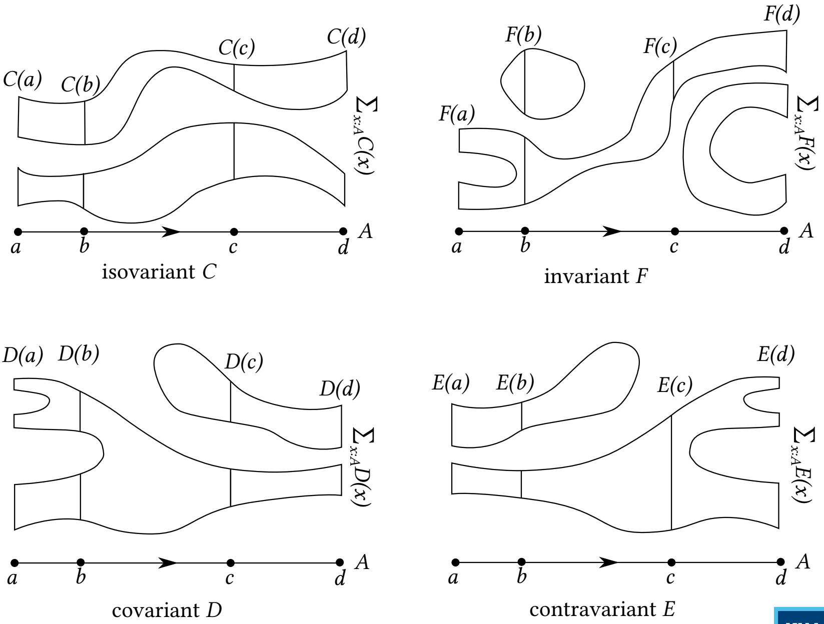
Thesis in pdf:

`http://people.cs.kuleuven.be/~dominique.devriese/ThesisAndreasNuyts.pdf`

Contact:

`andreasnuyts[at]gmail.com`

Questions? Opinions? Suggestions?



C(a) C(b) C(c) C(d)
$\Sigma_{x:A} C(x)$
a b c d A
isovariant C
F(a) F(b) F(c) F(d)
$\Sigma_{x:A} F(x)$
a b c d A
invariant F
D(a) D(b) D(c) D(d)
$\Sigma_{x:A} D(x)$
a b c d A
covariant D
E(a) E(b) E(c) E(d)
$\Sigma_{x:A} E(x)$
a b c d A
contravariant E

Higher variance

$$(A \xrightarrow{+} B) \xrightarrow{-} (A^{\mathbf{op}} \xrightarrow{+} B^{\mathbf{op}})$$

$X \mapsto X^{\mathbf{op}}$ has variance $+ - + + + \ldots$

Typical variances:

- $+ \equiv + + + + + + \ldots$,
- $- \equiv - + + + + \ldots$,
- $\times$: unclear (structure on zigzags?),
- $=$: doesn't matter (equality types are groupoids).

KU LEUVEN