Towards a directed HoTT based on 4 kinds of variance

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Directed HoTT: What and why?

<table>
<thead>
<tr>
<th>HoTT</th>
<th>$\infty$-groupoids</th>
<th>Homotopy Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>type $A$</td>
<td>$\infty$-groupoid $A$</td>
<td>space $A$</td>
</tr>
<tr>
<td>element $a : A$</td>
<td>object $a \in \text{obj}(A)$</td>
<td>point $a \in A$</td>
</tr>
<tr>
<td>$p : a =_A b$</td>
<td>isomorphism $p : a \to b$</td>
<td>path from $a$ to $b$</td>
</tr>
<tr>
<td>function $A \to B$</td>
<td>functor $A \to B$</td>
<td>continuous map $A \to B$</td>
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Directed HoTT

<table>
<thead>
<tr>
<th>$\varphi : a \simto_A b$</th>
<th>$(\omega, \omega)$-categories</th>
<th>Directed Homot. Theory</th>
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</thead>
<tbody>
<tr>
<td>morphism $\varphi : a \to b$</td>
<td>directed path from $a$ to $b$</td>
<td></td>
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</table>

Why?

- Math: Constructive theory for $(\omega, \omega)$-categories,
- Math: Domain-specific morphism is type-theoretic morphism,
- Progr: Automatic implementation of $f\text{map}$ for functions (but: canonicity problem),
- Bonus: Constructive (co)limits.
The approach

- Knowledge from (higher) category theory was not central in the development.
- Made sense of variance for dependent functions,
- Investigated the variance of important dependencies in HoTT,
- Adapted foundations of HoTT so as to incorporate functoriality from the start,
- Generalized univalence to give meaning to paths & morphisms,
- Inductive definition of morphisms with induction principle turns out to be possible.

**Work in progress:** Still a bit shabby.

- Semantics/consistency?
- Mysterious extra structure on types (zigzags?)
Directed:
- For all $a, b : A$, we can speak of morphisms $a \rightsquigarrow b$.

2-dimensional:
- Distinct elements $a, b : A$,
- Distinct morphisms $\varphi, \chi : a \rightsquigarrow b$,
- Either $\varphi \equiv \chi$ or not.

New: Co/Contravariance of assumptions ($x : \pm A$):

$$
\begin{array}{c}
\Gamma, x : \neg A \vdash B[x] \textbf{type} \\
\hline
\Gamma \vdash \prod_{x : A} B[x] \textbf{type}
\end{array}
$$

Opportunities for improvement:
- 2DTT excludes identity type. E.g. let $a = \Box$ be \textbf{covariant}.
  $\varphi : a \rightsquigarrow b$ gives $\varphi_* : (a = a) \to (a = b)$,
  so $\varphi_*(\text{refl } a) : a = b$.
  $a = \Box$ should be \textbf{invariant}.
- Decouple variance in types and in elements.
  2DTT: $\prod_{a : A} C(a)$ \textbf{contravariant} in $A$.
  Therefore: $C(a)$ and $f(a)$.
  Remarkably: $\prod^+$ that is \textbf{covariant} in \textit{domain}.
Meaning of structure: univalence

In HoTT:

- 2 structures: paths and functions.
- Univalence: path in universe is invertible function:
  \((A =_U B) \simeq (A \simeq B)\).

In directed HoTT

- 3 structures: paths, morphisms and covariant functions.
- Directed univalence: morphism in universe is covariant function:
  \((A \mapsto_U B) \simeq (A \mapsto B)\).
- Categorical univalence: path in any type is invertible morphism:
  \((a =_A b) \simeq (a \cong_A b)\).
  (Cfr. precategory vs. category)
At least 4 kinds of variance appear in HoTT:

Covariant: \( f : A \xrightarrow{\oplus} C \)
\( a \sim_A b \) implies \( f(a) \sim_C f(b) \),

Contravariant: \( f : A \xrightarrow{-} C \)
\( a \sim_A b \) implies \( f(a) \sim_C f(b) \),

Invariant: \( f : A \xrightarrow{\times} C \)
\( a \sim_A b \) implies (almost) nothing,

Isovariant: \( f : A \xrightarrow{=} C \)
\( a \sim_A b \) implies \( f(a) =_C f(b) \).

All functions preserve equality (\( \sim \) isomorphism), as in HoTT.
Meaning of variance for dept. functions

Tool: inductive type families for heterogenous equality/morphisms.

Let $C : A \rightarrow^\nu U$.

Any $f$ yields
\[
\frac{f(a) =_C f(b)}{p : a =_A b},
\]

Isovariant $f$ yields
\[
\frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b},
\]

Covariant $f$ yields
\[
\frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b},
\]

Contravariant $f$ yields
\[
\frac{f(a) \leftarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}.
\]

Whenever $p : a =_A b$,
\[
p_* : C(a) \simeq C(b),
\]

\[
\left( \frac{c =_C d}{p : a =_A b} \right) \simeq \left( p_*(c) =_{C(b)} d \right).
\]

Meaning depends on variance of $C$. 
Meaning of variance for dept. functions

Tool: inductive type families for heterogenous equality/morphisms.

Let $C : A \to^\nu U$.

- Any $f$ yields $f(a) =_C f(b)$, 

- Isovariant $f$ yields $f(a) =_C f(b)$, 

- Covariant $f$ yields $f(a) \sim^C f(b)$, 

- Contravariant $f$ yields $f(a) \sim^C f(b)$.

Meaning depends on variance of $C$.

Assume $C(x)$ covariant in $x$.

Whenever $\varphi : a \sim_A b$,

$$\varphi^* : C(a) \overset{\perp}{\to} C(b),$$

$$\left( \frac{c =_C d}{\varphi : a \sim_A b} \right) \sim \left( \varphi^*(c) =_C (b \ d) \right).$$
Meaning of variance for dept. functions

Tool: inductive type families for heterogenous equality/morphisms.
Let $C : A \to^\nu U$.

Any $f$ yields $\frac{f(a) =_{C} f(b)}{p : a =_A b}$,

Isovariant $f$ yields $\frac{f(a) =_{C} f(b)}{\phi : a \iso_A b}$,

Covariant $f$ yields $\frac{f(a) \iso_A f(b)}{\phi : a \iso_A b}$,

Contravariant $f$ yields $\frac{f(a) \iso_C f(b)}{\phi : a \iso_A b}$.

Meaning depends on variance of $C$.

Assume $C(x)$ contravariant in $x$.
Whenever $\phi : a \iso_A b$,

$\phi^* : C(b) \to C(a)$,

$\left( \frac{c =_{C} d}{\phi : a \iso_A b} \right) \iso \left( c =_{C(a)} \phi^*(d) \right)$.
Meaning of variance for dept. functions

Tool: inductive type families for heterogenous equality/morphisms.
Let $C : A \rightarrow^\nu U$.

Any $f$ yields $\frac{f(a) =_C f(b)}{p : a =_A b}$.

Isovariant $f$ yields $\frac{f(a) =_C f(b)}{\varphi : a \rightsquigarrow_A b}$.

Covariant $f$ yields $\frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}$.

Contravariant $f$ yields $\frac{f(a) \leftarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}$.

Meaning depends on variance of $C$.

Assume $C(x)$ isovariant in $x$.
Whenever $\varphi : a \rightsquigarrow_A b$,

$\varphi_* : C(a) \simeq C(b)$,

$\left(\frac{c =_C d}{\varphi : a \rightsquigarrow_A b}\right) \simeq \left(\varphi_*(c) =_C (b) d\right)$.
Meaning of variance for dept. functions

Tool: inductive type families for heterogenious equality/morphisms. Let $C : A \to U$.

- Any $f$ yields $\frac{f(a) = C f(b)}{p : a =_A b}$.
- Isovariant $f$ yields $\frac{f(a) = C f(b)}{\varphi : a \rightsquigarrow_A b}$.
- Covariant $f$ yields $\frac{f(a) \rightsquigarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}$.
- Contravariant $f$ yields $\frac{f(a) \leftarrow_C f(b)}{\varphi : a \rightsquigarrow_A b}$.

Meaning depends on variance of $C$.

When $C(x)$ invariant in $x$:

- Heterogeneous types are still defined and often have a meaning.

Invariant type families map morphisms to relations?

Invariant functions map morphisms to

- spans: $a \leftrightarrow * \to b$?
- cospans: $a \to * \leftarrow b$?
- zigzags:
  $a \to * \leftarrow \ldots \to * \leftarrow b$?
Meaning of variance for dept. functions

Tool: inductive type families for heterogenous equality/morphisms.

Let $C : A \rightarrow U$.

Any $f$ yields
\[ f(a) =_C f(b), \]
\[ p : a =_A b, \]

Isovariant $f$ yields
\[ f(a) =_C f(b), \]
\[ \phi : a \sim_A b, \]

Covariant $f$ yields
\[ f(a) \sim_C f(b), \]
\[ \phi : a \sim_A b, \]

Contravariant $f$ yields
\[ f(a) \sim_A f(b), \]
\[ \phi : a \sim_A b. \]

Meaning depends on variance of $C$.

Example:

$C : (X : U) \mapsto (X \dashv X)$.

Given morphism $h : A \dashv B$,

\[
\begin{align*}
(A \dashv A) & \overset{\sim}{\to} (A \dashv B) \overset{\sim}{\leftarrow} (B \dashv B) \\
\text{(f =}_C g dua(h) : A \sim_B) & \sim (h \circ f = g \circ h).
\end{align*}
\]

Commutative diagram = path along morphism

Commutative diagram for every morphism? Isovariance. E.g. $\text{id}_X$. 

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Example: a cospan type

Idea: \((a \vec{\wedge} b) \simeq \sum_{c:A} (a \rightsquigarrow c) \times (b \rightsquigarrow c)\)

Inductive type family

Cospan type family \(a \vec{\wedge}_A b\) has constructors

- \(\mathbf{triv} : \prod^+_a a \vec{\wedge} a\),
- Contravariance in \(a\): \((a' \rightsquigarrow a) \rightarrow\rightarrow (a \vec{\wedge} b) \rightarrow\rightarrow (a' \vec{\wedge} b)\),
- Contravariance in \(b\): \((b' \rightsquigarrow b) \rightarrow\rightarrow (a \vec{\wedge} b) \rightarrow\rightarrow (a \vec{\wedge} b')\).

To define \(f : \prod^?_{a,b:A} \prod^+_w a \vec{\wedge} b \rightarrow\rightarrow C(a, b, w)\), you need:

- \(f_{\mathbf{triv}} : \prod^+_a C(a, a, \mathbf{triv} a)\),
- \(C(a, b, w)\) contravariant in \(a\) and \(b\).

Computation:

- \(f(a, a, \mathbf{triv} a) \equiv f_{\mathbf{triv}}(a)\),
- \(f\) is isovariant in \(a\) and \(b\).
The morphism type

Inductive type family

Morphism type family $a \rightsquigarrow_{A} b$ has constructors

- **id** : $\prod_{a:A} a \rightsquigarrow a$,
- Contravariance in $a$: $(a' \rightsquigarrow a) \xrightarrow{\Downarrow} (a \rightsquigarrow b) \xrightarrow{\Downarrow} (a' \rightsquigarrow b)$,
- Covariance in $b$: $(b \rightsquigarrow b') \xrightarrow{\Downarrow} (a \rightsquigarrow b) \xrightarrow{\Downarrow} (a \rightsquigarrow b')$.

To define $f : \prod^{?}_{a,b:A} \prod^{+}_{\varphi : a \rightsquigarrow b} C(a, b, \varphi)$, you need:

- $f_{id} : \prod_{a:A} C(a, a, \text{id }a)$,
- $C(a, b, \varphi)$ contravariant in $a$ and covariant in $b$.

Computation:

- $f(a, a, \text{id }a) \equiv f_{id}(a)$,
- $f$ is isovariant in $a$ and $b$. 

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The identity type

**Inductive type family**

Identity type family $a =_A b$ has constructors

- $\text{refl} : \prod_{a:A} a = a$,
- Invariance in $a$ and $b$.

Although invariance cannot be written as a constructor, the scheme seems to generalize:

To define $f : \prod_{a,b:A} \prod_{p:a=b} \vdash C(a, b, p)$, you need:

- $f_{\text{refl}} : \prod_{a:A} C(a, a, \text{refl} a)$,
- $C(a, b, p)$ invariant in $a$ and $b$ (this condition is void).

Computation:

- $f(a, a, \text{refl} a) \equiv f_{\text{refl}}(a)$,
- $f$ is isovariant in $a$ and $b$. 
**Isovariance of \textbf{refl} and \textbf{id}**

- **\textbf{id} a** is isovariant in \(a\), because \(\frac{\text{id}_a = \text{id}_b}{\varphi: a \sim b}\) is equivalent to commutativity of:

\[
\begin{array}{c}
\text{id}_a \\
\downarrow \\
\text{id}_b \\
\end{array}
\begin{array}{c}
a \\
\varphi \\
b \\
\end{array}
\begin{array}{c}
a \\
\varphi \\
b \\
\end{array}
\begin{array}{c}
a \\
\varphi \\
b \\
\end{array}
\]

- **\textbf{refl} a** is isovariant in \(a\), because (interestingly!) \(\frac{\text{refl}_a = \text{refl}_b}{\varphi: a \sim b}\) is equivalent to commutativity of:

\[
\begin{array}{c}
\text{refl}_a \\
\| \\
\text{refl}_b \\
\end{array}
\begin{array}{c}
a \\
\varphi \\
b \\
\end{array}
\begin{array}{c}
a \\
\varphi \\
b \\
\end{array}
\begin{array}{c}
a \\
\varphi \\
b \\
\end{array}
\]

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The mystery of invariance, zigzags and relations

Questions:

- Are zigzags the way to go?
- What are morphisms between zigzags?
- Well-behaved type-theoretic account of zigzags?
  - Must not prove $\zeta^+ \circ \zeta = \text{id}$.
- Analogue of opposite for invariance?
  
  \[
  (A \xrightarrow{\rightarrow} B) \cong (A^{\text{op}} \xrightarrow{\rightarrow} B) \cong (A \xrightarrow{\rightarrow} B^{\text{op}}),
  \]
  \[
  (A \xrightarrow{\equiv} B) \cong (A^{\text{loc}} \xrightarrow{\rightarrow} B) \cong (A \xrightarrow{\rightarrow} B^{\text{core}}),
  \]
  \[
  f : (A^{\text{core}} \xrightarrow{\rightarrow} B) \text{ is function } A \rightarrow B \text{ that discards morphisms completely.}
  \]
  \[
  g : (A \xrightarrow{\rightarrow} B^{\text{loc}}) \text{ is function } A \rightarrow B \text{ that maps all structure to zigzags.}
  \]
  Can we have a core with degenerate/infinitesimal zigzags?

...
Directed function extensionality: Morphism of functions is a natural transformation,

\[(\prod_{a:A} C(a)) = (\lim_{a:A} C(a))\] for covariant \(C,\)

\[(\sum_{a:A} C(a)) = (\lim_{a:A} C(a))\] for covariant \(C,\)

Some functions, such as \(A \mapsto A^{\text{op}},\) have complicated ‘higher’ variance.
Thanks!

Thesis in pdf:
http://people.cs.kuleuven.be/~dominique.devriese/
ThesisAndreasNuyts.pdf

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Questions? Opinions? Suggestions?
Towards a directed HoTT based on 4 kinds of variance

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Higher variance

\((A \xrightarrow{\top} B) \xrightarrow{\sim} (A^{\text{op}} \xrightarrow{\top} B^{\text{op}})\)

\(X \mapsto X^{\text{op}}\) has variance \(+\)\(-\)\(+\)\(+\)\(+\)\(\ldots\).

Typical variances:

- \(+\Xi++\ldots\),
- \(-\Xi-+\ldots\),
- \(\times\): unclear (structure on zigzags?),
- \(\equiv\): doesn’t matter (equality types are groupoids).