Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

29 June 2015

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the *sheafification* process [MM92].

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the *sheafification* process [MM92].

Then (Grothendieck) sheafification has been extended to higher topos theory [Lur09].

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the *sheafification* process [MM92].

Then (Grothendieck) sheafification has been extended to higher topos theory [Lur09].

We will present here a work-in-progress attempt to define an homotopy type theoretic version of this process.

Let's recall that in a topos, a Lawvere-Tierney topology is an idempotent map $\Omega \to \Omega$, preserving true and products. We notice that it corresponds to a left-exact modality on the subobject classifier Ω.

Then, the sheafification process extend this modality to the whole topos.

Let's recall that in a topos, a Lawvere-Tierney topology is an idempotent map $\Omega \to \Omega$, preserving true and products. We notice that it corresponds to a left-exact modality on the subobject classifier Ω.

Then, the sheafification process extend this modality to the whole topos.

We want to follow this idea : from a left exact modality on HProp, we will define a left exact modality on all (finite) homotopy levels, by induction on this level.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Recall : Modalities

We use the same notion of modalities as in [Uni13, Section 7.7], but restricted to be on n-truncated types.

Definition

Let $n \geq -1$ be a truncation index. A left exact modality at level n is the data of

(i) *A predicate $P : \mathrm{Type}_n \to \mathrm{HProp}$*

(ii) *For every n-truncated type A, a n-truncated type $\bigcirc A$ such that $P(\bigcirc A)$*

(iii) *For every n-truncated type A, a map $\eta_A : A \to \bigcirc A$*

such that

(iv) *For every n-truncated types A and B, if $P(B)$ then*

$$\begin{cases} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{cases}$$

is an equivalence.

(v) for any $A : \text{Type}_n$ and $B : A \to \text{Type}_n$ such that $P(A)$ and $\prod_{x:A} P(Bx)$, then $P\left(\sum_{x:A} B(x)\right)$

(vi) for any $A : \text{Type}_n$ and $x, y : A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.

Conditions (i) to (iv) define a *reflective subuniverse*, (i) to (v) a *modality*.

Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos $\mathcal{T}$, with subobject classifier Ω.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos $\mathcal{T}$, with subobject classifier Ω.

$$\begin{array}{ccc} T & \xrightarrow{\{\cdot\}_T} & \Omega^T \\ & & \big\downarrow{\scriptstyle j^T} \\ & & (\Omega_j)^T \end{array}$$

Send T to Ω^T via the singleton map, then postcompose with $j : \Omega \to \Omega_j$

Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos $\mathcal{T}$, with subobject classifier Ω.

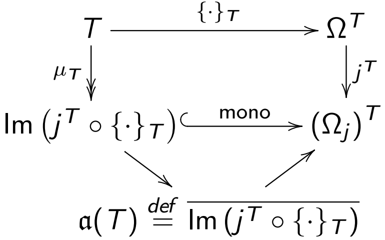
$$
\begin{array}{ccc}
T & \xrightarrow{\quad \{\cdot\}_T \quad} & \Omega^T \\
\mu_T \big\downarrow & & \big\downarrow j^T \\
\mathrm{Im}\left(j^T \circ \{\cdot\}_T\right) & \overset{\text{mono}}{\hookrightarrow} & \left(\Omega_j\right)^T
\end{array}
$$

Compute the image of this map: it is a subobject of $\left(\Omega_j\right)^T$

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Recall: Sheafification in topos

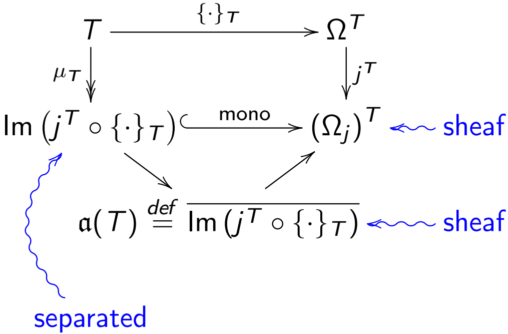
Let j be a Lawvere-Tierney topology on a topos $\mathcal{T}$, with subobject classifier Ω.



Close this subobject

Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos $\mathcal{T}$, with subobject classifier Ω.



Key points:

- $(\Omega_j)^T$ has to be a sheaf.
- A closed subobject of a sheaf should be a sheaf.

The predicate “is n-modal” on homotopy level n will be “is a Lawvere-Tierney n-sheaf”, and the required modality will be the n-sheafification.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Context

We work in homotopy type theory, i.e, Martin-Löf type theory, with univalence axiom (thus functional extensionality) and higher inductive types (although at the moment, we only need propositional truncation).

Context

Let $\bigcirc_{-1}$ be a left exact modality on HProp (homotopy level -1), $n \geqslant -1$ a truncation index, and $\bigcirc_n$ a left exact modality on n-Type (homotopy level n), coherent with $\bigcirc_{-1}$:

If T : HProp, then $\bigcirc_n T = \bigcirc_{-1} T$ where we still note T the image of T via the inclusion HProp $\hookrightarrow$ n-Type.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Questions

When generalizing construction in topos, several questions arises:

Questions

When generalizing construction in topos, several questions arises:

- Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects (embeddings) ?

Questions

When generalizing construction in topos, several questions arises:

- Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects (embeddings) ?
- The proof involves kernel pair of a surjection. How to generalize it ?

Questions

When generalizing construction in topos, several questions arises:

- Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects (embeddings) ?
- The proof involves kernel pair of a surjection. How to generalize it ?
- Do we use usual image, or a n-image arising from n-connected/n-truncated factorization system ?

Questions

When generalizing construction in topos, several questions arises:

- Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects (embeddings) ? Solved
- The proof involves kernel pair of a surjection. How to generalize it ? In progress
- Do we use usual image, or a n-image arising from n-connected/n-truncated factorization system ? Solved

Dense subobject I

Definition

Let E be a type. The closure *of a subobject of E with m-truncated homotopy fibers (or m-subobject of E, for short), classified by $\chi : E \to m\text{-Type}$, is the m-subobject of E classified by $\bigcirc_m \circ \chi$.*
An m-subobject of E classified by χ is said to be closed in *E if it is equal to its closure, i.e, $\chi = \bigcirc_m \circ \chi$.*

Practically, a m-subobject of E is just $\{e : E \ \& \ \chi \ e\}$, and its closure is $\{e : E \ \& \ \bigcirc_m (\chi \ e)\}$.

Dense subobject II

Definition

Let E be a type, and $\chi : E \to m\text{-Type}$. The m-subobject of E classified by χ is dense *in E when its $\bigcirc_m$-closure is equivalent to χ_E, i.e,*

$$\forall e : E,\ \ \bigcirc_m (\chi\ e) \simeq \mathbf{1}.$$

Practically, a m-subobject A of E is dense if, from the $\bigcirc_m$ point of view, you cannot make a difference between A and E.

Restriction

Definition

For any type E, characteristic map $\chi : E \to m\text{-Type}$ and $F : (n+1)\text{-Type}$, we define

$$\Phi_E^{\chi,m} : (E \to F) \to (\{e : E \ \&\ \chi\ e\} \to F)$$

as the map sending an arrow $f : E \to F$ to its restriction $f \circ \pi_1$.

Requirements

We want a predicate on $(n+1)$-Type, which we call *sheaf property*, satisfying:

- if $\bigcirc_n$ is the identity modality, then everybody should be a sheaf
- a closed (-1)-subobject of a sheaf should be a sheaf
- the type of modal n-Type should be a sheaf
- if T is a sheaf, then $X \to T$ should be a sheaf, for any X

Requirements

We want a predicate on $(n+1)\text{-}\mathrm{Type}$, which we call *sheaf property*, satisfying:

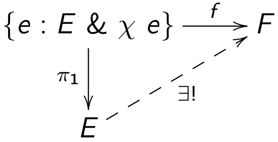
- if $\bigcirc_n$ is the identity modality, then everybody should be a sheaf
- a closed (-1)-subobject of a sheaf should be a sheaf
- the type of modal $n\text{-}\mathrm{Type}$ should be a sheaf
- if $T : X \to (n+1)\text{-}\mathrm{Type}$ such that any $T\,x$ is a sheaf, then $\prod_{x:X} T\,x$ should be a sheaf.

Sheaves

Following the topos-theoretic idea, we use:

Definition (Sheaves)

A type F *of* $(n+1)\text{-Type}$ *is a* $(n+1)$-sheaf *for any type* E *and all dense* (-1)*-subobject* $\chi : E \to (-1)\text{-Type}$, $\Phi_E^{\chi,-1}$ *is an equivalence. In other words, the dotted arrow exists and is unique.*

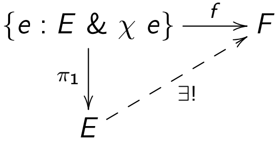


Sheaves

Following the topos-theoretic idea, we use:

Definition (Sheaves)

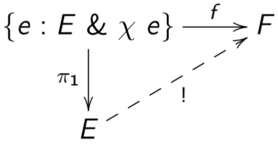
A type F of $(n+1)$-Type is a $(n+1)$-sheaf if it is separated, and for any type E and all dense (-1)-subobject $\chi : E \to (-1)\text{-Type}$, $\Phi_E^{\chi,-1}$ is an equivalence. In other words, the dotted arrow exists and is unique.



Separated type

Definition (Separated Type)

A type F in $(n+1)$-Type is separated *if for any type E, and all dense n-subobject $\chi : E \to n\text{-Type}$, $\Phi_E^{\chi,n}$ is an embedding. In other words, the dotted arrow, if exists, is unique.*



Two steps

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

We will proceed in two steps:

(i) *separation:* From any T in $(n+1)\text{-}\mathrm{Type}$, we construct its *free* separated object $\square_{n+1}\, T$.

(ii) *completion:* We add what is missing for the free separated type to be a sheaf by using closure.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Let $T : (n+1)\text{-Type}$. We define $\Box_{n+1}\, T$ as the image of $\bigcirc_n^T \circ \{\cdot\}_T$, as in

$$\begin{array}{ccc} T & \xrightarrow{\{\cdot\}_T} & n\text{-Type}^T \\ \Big\downarrow{\scriptstyle \mu_T} & & \Big\downarrow{\scriptstyle \bigcirc_n^T} \\ \Box_{n+1}\, T & \longrightarrow & \left(n\text{-Type}^{\bigcirc}\right)^T \end{array}$$

where $\{\cdot\}_T$ is the singleton map $\lambda(t : T),\ \lambda(t' : T),\ t = t'$.

Let $T : (n+1)\text{-Type}$. We define $\Box_{n+1}\, T$ as the image of $\bigcirc_n^T \circ \{\cdot\}_T$, as in

$$\begin{array}{ccc} T & \xrightarrow{\{\cdot\}_T} & n\text{-Type}^T \\ \downarrow{\scriptstyle \mu_T} & & \downarrow{\scriptstyle \bigcirc_n^T} \\ \Box_{n+1}\, T & \longrightarrow & \left(n\text{-Type}^{\bigcirc}\right)^T \end{array}$$

where $\{\cdot\}_T$ is the singleton map $\lambda(t : T),\ \lambda(t' : T),\ t = t'$.
$\Box_{n+1}\, T$ can be given explicitly by

$$\begin{aligned} \Box_{n+1}\, T &\overset{def}{=} \mathrm{Im}(\lambda\ t : T,\ \lambda\ t',\ \bigcirc_n (t = t')) \\ &\overset{def}{=} \sum_{u : T \to n\text{-Type}^{\bigcirc}} \left\| \sum_{a : T} (\lambda t,\ \bigcirc_n (a = t)) = u \right\| . \end{aligned}$$

At first, we prove that:

Proposition

For any $T : (n+1)\text{-Type}$, $\square_{n+1}\, T$ *is separated.*

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

At first, we prove that:

Proposition

For any $T : (n+1)\text{-Type}$, $\Box_{n+1}\ T$ *is separated.*

Then, we want

Theorem

$(\Box_{n+1}, \mu)$ *defines a modality on* $(n+1)\text{-Type}$.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Sketch of proof

In topoi, the proof goes this way:

Sketch of proof

In topoi, the proof goes this way:

- μ_T is a surjection, thus it coequalizes its kernel pair

$$T \times_{\Box_{n+1} T} T \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} T \xrightarrow{\mu_T} \Box_{n+1} T$$

Sketch of proof

In topoi, the proof goes this way:

- μ_T is a surjection, thus it coequalizes its kernel pair

$$T \times_{\Box_{n+1}\, T} T \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} T \xrightarrow{\mu_T} \Box_{n+1}\, T$$

- Then $T \times_{\Box_{n+1}\, T} T = \overline{\Delta}$, where $\Delta = \{(x, y) : T^2 \ \&\ x = y\}$. The following is a coequalizer

$$\overline{\Delta} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} T \xrightarrow{\mu_T} \Box_{n+1}\, T$$

Sketch of proof

Then, if Q is any separated type and $f : T \to Q$, it makes the diagram

$$\overline{\Delta} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} T \xrightarrow{f} Q$$

commute, thus f factors through $\square_{n+1} T$.

We would like to use the same idea, replacing the kernel pair by the Čech nerve.

At the moment, we only assumed as an axiom that surjections are colimits of their Čech nerves, seen as graphs. It allows us to finish the proof.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

For any T in $(n+1)$-Type, $\bigcirc_{n+1} T$ is defined as the closure of $\square_{n+1} T$, seen as a subobject of $T \to n$-$\mathrm{Type}^{\bigcirc}$. $\bigcirc_{n+1} T$ can be given explicitly by

$$\bigcirc_{n+1} T \overset{def}{=} \sum_{u : T \to n\text{-}\mathrm{Type}^{\bigcirc}} \bigcirc_{-1} \left\| \sum_{a : T} (\lambda t,\ \bigcirc_n (a = t)) = u \right\|.$$

As above, we first prove that:

Proposition

For any $T : (n+1)\text{-Type}$, $\bigcirc_{n+1} T$ *is a sheaf.*

It is true because of the requirement we asked about sheaves:

Lemma

Let $X : (n+1)\text{-Type}$ *and* U *be a sheaf. If* X *embeds in* U*, and is closed in* U*, then* X *is a sheaf.*

As above, we first prove that:

Proposition

For any $T : (n+1)\text{-Type}$, $\bigcirc_{n+1} T$ *is a sheaf.*

It is true because of the requirement we asked about sheaves:

Lemma

Let $X : (n+1)\text{-Type}$ *and* U *be a sheaf. If* X *embeds in* U*, and is closed in* U*, then* X *is a sheaf.*

Then:

Theorem

$(\bigcirc_{n+1}, \nu)$ *defines a left-exact modality.*

Sketch of proof

Let $T, Q : (n+1)\text{-}\mathrm{Type}$ such that Q is a sheaf. Let $f : T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\square_{n+1} f : \square_{n+1} T \to Q$.

Sketch of proof

Let $T, Q : (n+1)\text{-}\mathrm{Type}$ such that Q is a sheaf. Let $f : T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\Box_{n+1} f : \Box_{n+1} T \to Q$.

But as $\bigcirc_{n+1} T$ is the closure of $\Box_{n+1} T$, $\Box_{n+1} T$ is dense into $\bigcirc_{n+1} T$, so the sheaf property of Q allows to extend $\Box_{n+1} f$ to $\bigcirc_{n+1} f : \bigcirc_{n+1} T \to Q$.
As all these steps are universal, the composition is.

Sketch of proof

Let $T, Q : (n+1)\text{-Type}$ such that Q is a sheaf. Let $f : T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\Box_{n+1} f : \Box_{n+1} T \to Q$.

But as $\bigcirc_{n+1} T$ is the closure of $\Box_{n+1} T$, $\Box_{n+1} T$ is dense into $\bigcirc_{n+1} T$, so the sheaf property of Q allows to extend $\Box_{n+1} f$ to $\bigcirc_{n+1} f : \bigcirc_{n+1} T \to Q$.
As all these steps are universal, the composition is.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Starting from the left-exact modality $\bigcirc_{-1} P = \neg\neg P$, this allows us to build a model satisfying excluded middle for HProp, without axiom.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Starting from the left-exact modality $\bigcirc_{-1} P = \neg\neg P$, this allows us to build a model satisfying excluded middle for HProp, without axiom.

With the same modality $\neg\neg$, we hope to be able to formalize the proof of independance of continuum hypothesis (actually, just the consistance of $\neg\mathrm{HC}$).

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Universes

The construction can be written inductively:

- $\bigcirc : \forall\ (n : nat),\ n\text{-Type} \to n\text{-Type}$
- $\bigcirc_{-1}$ is a left exact modality on HProp
- $\bigcirc_{n+1} \stackrel{def}{=} \lambda T : (n+1)\text{-Type},$

$$\sum_{u : T \to n\text{-Type}^{\bigcirc}} \bigcirc_{-1} \left\| \sum_{a : T} u = (\lambda t,\ \bigcirc_n (a = t)) \right\|$$

Here , the universe level increases strictly at each step, hence it is impossible to take the fixpoint: we would need universes to be indexed by (non-finite) ordinals.

Čech nerve

The main step to finish the construction is to define Čech nerve in HoTT, as well as the computation of their colimits.

We will rather try to define general simplicial objects.

Kevin Quirin and Nicolas Tabareau
Inria, Mines de Nantes
Nantes, France

Simplicial types

Hugo Herbelin [Her14] gives an inductive definition of semi-simplicial types, which can probably be adapted to define simplicial types, but is quite unusable for n-types with $n \geqslant 4$.

Homotopy type system

One idea is to use homotopy type system, introduced by V.V., to see Type as a model category. Then, we should be able to formalize homotopy colimits in type theory.

P.J. Cohen and M. Davis, *Set theory and the continuum hypothesis*, WA Benjamin New York, 1966.

Hugo Herbelin, *A dependently-typed construction of semi-simplicial types*, Mathematical Structures in Computer Science (2014), to appear.

Jacob Lurie, *Higher topos theory*, Annals of mathematics studies, Princeton University Press, Princeton, N.J., Oxford, 2009.

Saunders MacLane and Ieke Moerdijk, *Sheaves in geometry and logic*, Springer-Verlag, 1992.

Univalent Foundations Project, *Homotopy type theory: Univalent foundations for mathematics*, `http://homotopytypetheory.org/book`, 2013.