

The geometry of constancy

(in HoTT and in `cubicaltt`)

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Exiting propositional truncations

Often we have $\|X\| \rightarrow X$, even when we don't know whether X is empty or inhabited.

E.g. For any $f : \mathbb{N} \rightarrow \mathbb{N}$, we have $\|\sum_{n:\mathbb{N}} f n = 0\| \rightarrow \sum_{n:\mathbb{N}} f n = 0$.

If there is a root of f , then we can find one.

Exiting propositional truncations

However, global choice

$$\prod_{X:U} \|X\| \rightarrow X$$

implies that all types have decidable equality.

(And even $X + \neg X$ for all $X : U$ if we have quotients.)

By Hedberg's theorem, if every type has decidable equality then every type is a set and hence global choice negates univalence.

So global choice is both a constructive and a homotopy type theory taboo.

Exiting propositional truncations

Theorem (with Nicolai, Thierry and Thorsten):

A type X has a **choice** function $\|X\| \rightarrow X$ iff it has a **constant** endomap $X \rightarrow X$.

Question:

*Can we eliminate $\|X\| \rightarrow A$ using a **constant** map $X \rightarrow A$?*

Two answers: Yes (Nicolai Kraus) and no (Mike Shulman).

Nicolai considers **coherently** constant functions.

Mike considers **arbitrary** constant functions.

Constancy

There are many notions of constancy.

We investigate the following:

1. A function $f : X \rightarrow A$ is constant if any two of its values are equal.

$$\text{constant } f \stackrel{\text{def}}{=} \prod_{x,y:X} fx = fy.$$

2. This is **data** rather than property, unless A is a set.

Called a **modulus of constancy** of f .

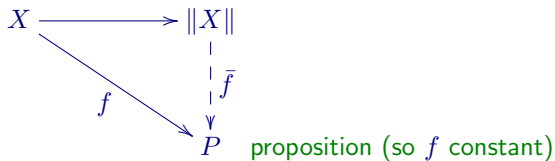
A function can have zero, one or more moduli of constancy.

3. E.g. the function $f : 1 \rightarrow S^1$ with definitional value **base** has \mathbb{Z} -many moduli of constancy $\kappa_n : \text{constant } f$:

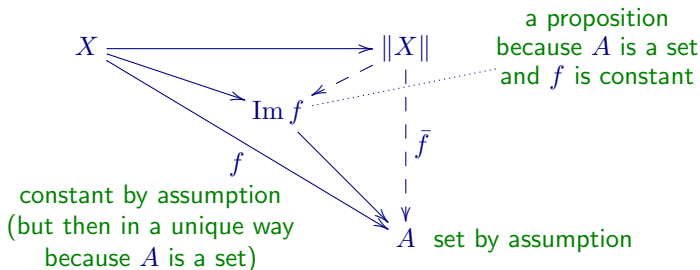
$$\kappa_n(x)(y) \stackrel{\text{def}}{=} \text{loop}^n.$$

Set-valued constant functions

1. For any proposition P , by definition of truncation:

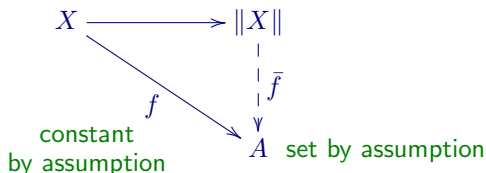


2. Can replace P by a set A :

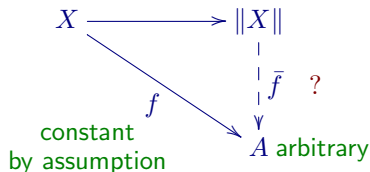


Propositional truncation as a set quotient

1. I.e. $\|X\|$ is the set-quotient of X by the chaotic relation:

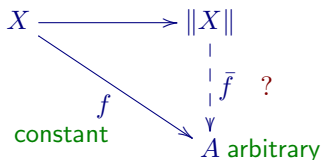


2. Can we replace A by an arbitrary type?



No, not in general (Shulman, <http://homotopytypetheory.org/2015/06/11/not-every-weakly-constant-function-is-conditionally-constant/>)

When do we get a factorization of a constant function?



The factorization is possible if any of the following conditions holds:

1. X is empty.
2. X has a given point.
3. We have a function $\|X\| \rightarrow X$.
4. We have a function $A \rightarrow X$.
5. A is a set.

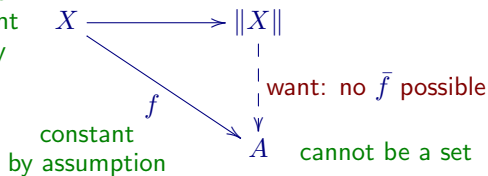
What other sufficient conditions?

And what about necessary conditions?

Also, given any factorization, we can construct another one for which the triangle commutes judgementally.

How to construct a counter example

cannot have
a known point
or be empty



Natural attempt to get a counter-example

Let $s : S^1$ be an arbitrary point of the circle.

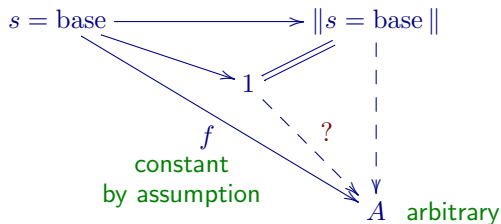
Let A be an arbitrary type.

Let $f : s = \text{base} \rightarrow A$ be constant.

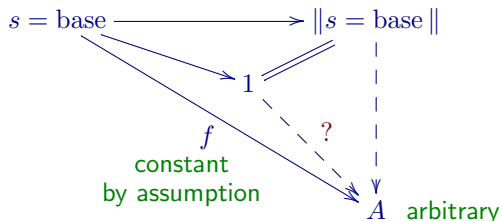
We can't know a point of the path space $s = \text{base}$ in general.

But we know it is inhabited, that is, $\|s = \text{base}\|$

Hence $\|s = \text{base}\| = 1$ by propositional univalence/extensionality.



Attempt to get a counter-example

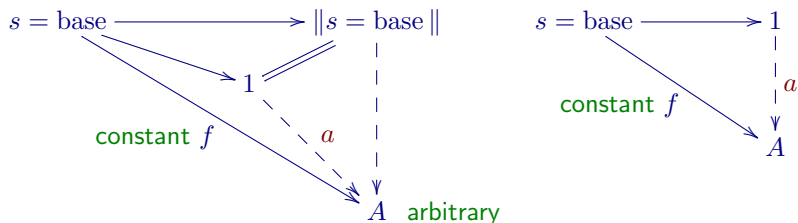


Can we expect to be able to get a point of an arbitrary type A , from any given constant function $f : s = \text{base} \rightarrow A$, even though we can't expect to get a point of $s = \text{base}$ in general?

To our surprise, we can.

The attempt fails.

Theorem/Construction



For any $s : S^1$ and any constant function $f : s = \text{base} \rightarrow A$ into an arbitrary type, we can find $a : A$ such that $fp = a$ for all $p : s = \text{base}$.

$$\prod_{s:S^1} \prod_{A:U} \prod_{f:s=\text{base} \rightarrow A} \text{constant } f \rightarrow \sum_{a:A} \prod_{p:s=\text{base}} fp = a.$$

Proof outline

1. First show that for any given family of constant functions

$$f : \prod_{s:S^1} s = \text{base} \rightarrow A(s),$$

each of them factors through 1. We get $\bar{f} : \prod_{s:S^1} A(s)$

This allows us to use induction on the circle and on paths.

2. For any type X , consider the universal constant map on X , $\beta_X : X \rightarrow S(X)$, constructed as a HIT.
3. By (1) applied to the family $\beta_s : s = \text{base} \rightarrow S(s = \text{base})$ given by (2), we get a function $\bar{\beta} : \prod_{s:S^1} S(s = \text{base})$.
4. Now, given a single constant function $f : s = \text{base} \rightarrow A$, it factors through the universal constant map $\beta_s : s = \text{base} \rightarrow S(s = \text{base})$ as $f' : S(s = \text{base}) \rightarrow A$ by (2), and hence we get the required point of A as using (3), as $f'(\bar{\beta}(s))$.

Step 1

For any $f : \prod_{s:S^1} s = \text{base} \rightarrow A(s)$, with f base constant, there is $\bar{f} : \prod_{s:S^1} A(s)$ such that $f s p = \bar{f} s$ for all $p : s = \text{base}$.

1. **Lemma** Any transport of a value of f is a value of \bar{f} :

$$\prod_{b,b':S^1} \prod_{r:b=b} \prod_{l:b=b'} \sum_{q:b'=b} \text{transport } l(f b r) = \bar{f} b' q.$$

This doesn't depend on the fact that S^1 is the circle or on the constancy of f base, and has a direct proof by based path induction.

2. We are interested in this particular case:

$$\sum_{q:\text{base}=\text{base}} \text{transport loop } (f \text{ base } (\text{refl base})) = \bar{f} \text{ base } q.$$

3. Then the constancy of f base gives

$$\text{transport loop } (f \text{ base } (\text{refl base})) = \bar{f} \text{ base } (\text{refl base}),$$

which makes S^1 -induction work.

Step 2

For any type X , consider the universal constant map on X ,

$$\beta : X \rightarrow S(X),$$

defined as a HIT with higher constructor

$$\ell : \prod_{x,y:X} \beta x = \beta y.$$

$$\begin{array}{ccc} X & \xrightarrow{\beta} & SX \\ & \searrow \text{constant } f & \downarrow \bar{f} \text{ such that } \text{ap } \bar{f}(\ell xy) = kxy \\ & & A \end{array}$$

with modulus $k : \prod_{x,y} fx = fy$

When X is the terminal type 1 , we get the circle S^1 .

Universal property of the constancy HIT

$$\begin{aligned}\beta & : X \rightarrow S(X), \\ \ell & : \prod_{x,y:X} \beta x = \beta y.\end{aligned}$$

There is an equivalence

$$\begin{aligned}SX \rightarrow A & \cong \sum_{f:X \rightarrow A} \text{constant } f \\ g & \mapsto (g \circ \beta, \lambda xy. \text{ap } g (\ell xy)).\end{aligned}$$

This generalizes the universal property of the circle

$$\begin{aligned}S^1 \rightarrow A & \cong \sum_{a:A} a = a \\ & \cong \sum_{f:1 \rightarrow A} \text{constant } f.\end{aligned}$$

Side remark

(Not used in the proof, at least not explicitly.)

1. The universal constant map $\beta_X : X \rightarrow S(X)$ is a surjection.
2. The type $S(X)$ is conditionally connected, meaning that

$$\prod_{s,t:S(X)} \|s = t\|.$$

(“Conditionally” because it is empty if (and only if) X is empty.)

cubicaltt proof

Demonstrate and discuss some fragments of the `geometryOfConstancy.ctt` file (on my papers web page).

The constant factorization problem

Because the universal map $X \rightarrow \|X\|$ into a proposition is constant (in a unique way), the universal property of $S(X)$ gives a function

$$\prod_{X:U} S(X) \rightarrow \|X\|.$$

The existence of a function in the other direction,

$$\prod_{X:U} \|X\| \rightarrow S(X),$$

is equivalent to the statement that all constant functions $f : X \rightarrow A$ factor through $X \rightarrow \|X\|$.

But we know that this is not the case, by Shulman's construction.

However, this does hold for $X = (s = \text{base})$ and all A .

Step 3

By (1) applied to the family $\beta_s : s = \text{base} \rightarrow S(s = \text{base})$ of constant functions given by (2), we get a function

$$\bar{\beta} : \prod_{s:S^1} S(s = \text{base}).$$

This is perhaps surprising, because we don't have, of course,

$$\prod_{s:S^1} s = \text{base},$$

as that would mean that that the circle is contractible.

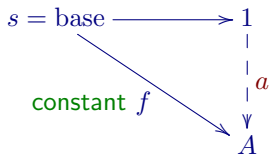
How come we are able to pick a point of the generalized circle $S(s = \text{base})$, without being able to pick a point of the path space $s = \text{base}$, naturally in $s : S^1$?

Step 4

Now, given a **single** constant function $f : s = \text{base} \rightarrow A$, it factors through the universal constant map $\beta_s : s = \text{base} \rightarrow S(s = \text{base})$ as $f' : S(s = \text{base}) \rightarrow A$ by (2), and hence we get the required point of A using (3), as

$$a \stackrel{\text{def}}{=} f'(\bar{\beta}(s)).$$

Theorem



Conjecture

In a type theory with $\| - \|$ and (hence) function extensionality.

All constant functions $f : X \rightarrow A$ of any two types factor through $X \rightarrow \|X\|$ if and only if all types are sets (zero-truncated).

And hence univalence fails if all constant functions factor through the truncations of their domains.

(Shulman's construction exhibits a family of constant functions such that if all of them factor through the truncation of their domain, then univalence fails.)