Towards a Directed HoTT with Four Kinds of Variance

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May 15, 2015

Homotopy type theory (HoTT) offers a constructive way of working with ∞-groupoids. When we use it as a foundation for mathematics, it yields the useful principle 'isomorphism implies equality.' When the canonicity problem of HoTT is solved, it would allow programmers to identify equivalent types A and B, then apply a functor F to them, and then turn the identification $FA =_{\mathcal{U}} FB$ again into an automatically defined equivalence.

A directed formulation of HoTT, in which we do not only have isomorphisms $a =_A b$, but also unidirectional morphisms $a \rightsquigarrow_A b$, extends the aforementioned applications. We would obtain a constructive way of working with higher categories. When we use it as a foundation for mathematics, it might yield principles as 'a group morphism from G to H implies a morphism $G \rightsquigarrow_{\mathrm{Grp}} H$.' When the canonicity problem is solved, it would allow programmers to turn a function $A \to B$ into a morphism, then apply a functor F, and then turn the morphism $FA \rightsquigarrow_{\mathcal{U}} FB$ again into an automatically defined function.

In [2DTT], D. Licata and R. Harper give a formulation for directed type theory in two dimensions, however at the cost of abolishing the identity type family. I propose a few modifications to the formal type system of HoTT, most importantly the introduction of 2 additional kinds of variance for functions, to obtain an infinite dimensional directed type theory which has both identity and morphism types. Invariant functions discard morphisms and allow us to keep the identity type. Isovariant functions map morphisms to paths; the dependent ones allow us to reason succinctly about commuting diagrams. Both morphism and identity type have an inductive definition and they differ only in their variance: whereas the identity type is invariant in its endpoints, the morphism type is contravariant in the source and covariant in the target. Interpreting the variance in source and target as two extra constructors, we arrive at the proper induction principle.

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Eliminating out of Truncations

Talk abstract, April 2015

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If we want to perform a construction or show a result which does not hold for types with non-trivial higher equality structure, we often choose to only work with n-types, for some suitable number $n \geq -1$. To give examples: for algebraic structures such as groups, we may require the type of elements to be a 0-type, and for categories, the type of objects has to be a 1-type, while one might want to do some form of "traditional logic" with (-1)-types. This way, we can avoid coherence problems that could potentially occur on higher levels that we may not even be interested in. The truncation operator $\|-\|_n$, which transforms any type A into an n-type $\|A\|_n$, can be viewed and implemented as a higher inductive type, but is certainly somewhat special. It is a modality (an idempotent monad in some appropriate sense), and it allows us to work completely in the "subuniverse" of n-types. This becomes difficult if, at some point, we need to leave this "subuniverse". The universal property of $\|-\|_n$ says that functions $(\|A\|_n \to B)$ correspond to functions $(A \to B)$, but only if B happens to be an n-type.

It may therefore be interesting to derive a more powerful "universal property" for $\|-\|_n$ which is not restricted to n-types B, but works for any m-type B. Here, m is a fixed number that may be anything greater than n, including ∞, in which case we do not put any restriction on B. Intuitively, what we need to do is to require the functions $(A \to B)$ to satisfy certain coherences if we want them to correspond to functions $(\|A\|_n \to B)$.

I will present an outline of my solution for the propositional truncation (arXiv:1411.2682), i.e. $n \equiv -1$, where (in the currently considered type theory) m is any number, but has to be fixed externally. This needs some specific "semi-simplicial type". I use the construction to illustrate that we might want a type theory that allows the construction of "Reedy-fibrant diagrams" and its limits (sometimes called "infinitary type theory"). Joint work with Paolo Capriotti and Andrea Vezzosi has further yielded a solution for the case $m \equiv n + 1$ (i.e. n is no longer required to be -1). I will try to explain why the remaining cases (general $n > -1$, arbitrary m greater than n) seem to be harder than the solved ones. Intuitively, this is because they combine two different kinds of coherence problems.

Lawvere-Tierney Sheafification in Homotopy Type Theory

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May 29, 2015

Abstract

Sheafification is a popular tool in topos theory which allows to extend the internal logic of a topos with new principles. One of its most famous applications is the possibility to transform a topos into a boolean topos using the dense topology, which corresponds in essence to Gödel's double negation translation. A computer-checked construction of Lawvere-Tierney sheafification in homotopy type theory would allow in particular to give a meaning to the (propositional) law of excluded middle inside homotopy type theory, being compatible with the full type-theoretic axiom of choice. We give here some key points of this construction.

Sheafification [MM92] is a very powerful geometric construction that has been initially stated in topology and has quickly been lifted to mathematical logic. In the field of topos theory, it provides a way to construct new toposes from already existing ones, allowing logical principles—that can not be proved to be true or false in the old topos—to be valid (or invalid) in the new topos. A famous application has been developed by Cohen [CD66] to prove that the continuum hypothesis is independent of the usual axioms of Zermelo-Frænkel set theory, even in presence of the axiom of choice (AC). The initial work of Cohen uses forcing but can be rephrased in terms of sheafification [MM92].

As the notion of higher toposes appears to correspond very closely to homotopy type theory, higher topos theory [Lur09] provides a new hope that tackling the problem of extending the power of homotopy type theory using sheafification is actually possible.

Lawvere-Tierney Sheafification in Topos Theory

Lawvere-Tierney sheafification in a topos $\mathcal{E}$ is based on an abstract point of view on the topology to be considered, being simply defined by an endomorphism on the classifying object Ω of $\mathcal{E}$

$$j : \Omega \to \Omega$$

that is required to preserve true (j true = true), to be idempotent ($j \circ j = j$) and compatible with products ($j \circ \wedge = \wedge \circ (j, j)$). A typical example is given by double negation.

Every Lawvere-Tierney topology j induces a closure operator $A \mapsto \overline{A}$ on subobjects. If we see a subobject A of E as a characteristic function χ_A, the closure $\overline{A}$ corresponds to the subobject of E whose characteristic function is

$$\chi_{\overline{A}} = j \circ \chi_A.$$

A subobject A of E is said to be dense when $\overline{A} = E$.

The idea is then to define sheaves in $\mathcal{E}$ as objects of $\mathcal{E}$ for which it is impossible to make a distinction between objects and their dense subobjects. This idea is formalized by saying that for every dense subobject A of E, the following canonical map is an isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(E, F) \to \mathrm{Hom}_{\mathcal{E}}(A, F). \tag{1}$$

One can show that $\mathrm{Sh}_{\mathcal{E}}$, the full sub-category of $\mathcal{E}$ given by sheaves, is again a topos, with classifying object

$$\Omega_j = \{P \in \mathcal{E} \mid jP = P\}.$$

Thus, in case of the double negation, the resulting topos is boolean and admits classical reasoning.

Furthermore, one can define a left adjoint to the inclusion, the sheafification functor

$$a_j : \mathcal{E} \to \mathrm{Sh}_{\mathcal{E}}$$

which exhibits $\mathrm{Sh}_{\mathcal{E}}$ as a reflective subcategory of $\mathcal{E}$ (which is a particular case of localization). This means that logical principles valid in $\mathcal{E}$ are still valid in $\mathrm{Sh}_{\mathcal{E}}$.

Overview of the Result

To extend Lawvere-Tierney sheafification to homotopy type theory, the first thing to understand is that the construction can not be done in one single step anymore. It must rather be performed by induction on the level of homotopy types. More precisely, the first layer of sheafification is defined for Type_0, given a topology on Type_{-1}. This part corresponds to sheafification for toposes. Then, assuming that the sheafification has been constructed up to level n, one can define the sheafification for Type_{n+1}.

This inductive step requires to formalize the notion of left-exact reflective subuniverse, which corresponds to a stratified version of left-exact modality introduced in [Uni13, Chapter 7]. A Lawvere-Tierney topology can thus be seen as a left-exact modality on Type_{-1} and sheafification as an inductive process that extends it as a left-exact modality on Type_n for any n.

Two steps keep us from finishing the construction:

- Sheafification in topos theory uses the property that epimorphisms are coequalizers of their kernel pairs. The translation of this in homotopy type theory is "Effective epimorphisms are exactly the surjections". The difficulty here is to give a meaning to "effective epimorphism", and thus to Čech nerve and (homotopy) colimits. When it is done, we hope that it will be easy to prove the needed fact.

- From a technical point of view, the handling of universes by Coq is at the moment not powerfull enough to allow us to formalize completely the construction. More precisely, sheafification relies on a function $\mathrm{Type}_n \to \mathrm{Type}_{n+1}$ increasing strictly universe levels, which we want to take the fixpoint. Thus, we need to allow universe levels to be non-finite ordinals.

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An inductive dependently-typed construction of simplicial sets and of similar presheaves over a Reedy category

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A Reedy category generalizes a kind of structure over which various common presheaves can be built, such as semi-simplicial sets, simplicial sets, cubical sets or even, though of much simpler structure, globular sets. Morphisms of a Reedy category are split into upwards and downwards morphisms and any arbitrary morphism of a Reedy category factors uniquely as the composition of a downwards and of an upwards morphism, in the same way as any morphism in, say, the simplex category Δ or in the cube category factors through a coface map and a codegeneracy map.

In a directed Reedy category, only upwards morphisms are present. The typical example of this is the subcategory Δ_i of injective morphisms of Δ from which semi-simplicial sets (also known as Δ-sets) are defined as a presheaf. In this context, Awodey and LeFanu Lumsdaine sketched the idea of an alternative inductively-defined dependently-typed construction of a semi-simplicial set. When formalized in the context of homotopy type theory, this even leads to a notion of semi-simplicial types, where types may have non-trivial homotopy levels, on the contrary of sets.

In a previous work, we proposed a precise definition of Awodey and LeFanu Lumsdaine's sketch of what a dependently-typed presentation of semi-simplicial types could look like within a type theory equipped with (at least) a strict equality. In the current work, we extend this construction to an inductively-defined dependently-typed construction of a presheaf over an arbitrary Reedy category of countable cardinal. In particular, this provides with an original construction of simplicial sets as well as cubical sets where face maps and degeneracy maps are intrinsically part of the structure of sets rather than axiomatized aside.

Unfortunately, the construction is pretty involved and at the current stage, only the inductively-defined dependently-typed definition of a "Reedy type" is given. Showing the correspondence with the presheaf definition, as well as defining morphisms, composition, products or exponentials would require another significant amount of complexity. Also, while our construction of semi-simplicial types was formally checked in Coq, we did not machine-checked our definition of Reedy types.

The standard definition of simplicial sets as families of sets equipped with face and degeneracy maps satisfying some appropriate equational theory does not ensure the decidability for a simplex of being degenerate or not. Contrastingly, our approach is constructive, in the sense that whether a simplex is degenerate or not is hard-wired in the definition and hence decidable. In particular, the correspondence with the presheaf definition holds only classically. Despite the degeneracies are hard-wired in the construction, we however foresee that building the exponential will still require classical reasoning in the general case.

Ordinary and Indexed W-Types

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Indexed W-types model inductive families, just as ordinary W-types model inductive types. In extensional type theory, indexed W-types were shown constructible from ordinary ones by Gambino and Hyland [3] using equivalent categorical terms: they construct initial algebras for dependent polynomial functors from non-dependent ones in the setting of a locally cartesian-closed category. In intensional type theory with function extensionality, an analogous result should hold when considering the corresponding homotopified notion [1] of (indexed) W-types.

Though tedious, this is provable using seemingly ad-hoc term-level manipulations following essentially the extensional ideas (we are aware of an account by Lumsdaine in terms of Coq code starting from strict ordinary W-types and using large elimination). Instead, we want to highlight a conceptually clean alternative suggested by the rolling rule [2] and cartesianess of the monad associated to any morphism via the codomain bifibration. This illuminates a deeper categorical nature of the extensional construction [3] and makes it amendable in a straightforward fashion to higher categorical generalization in terms of locally cartesian-closed quasi-categories (completed theorem).

Recent work by Szumiło [4] and Kapulkin (to be published) exhibits the syntax of intensional type theory with function extensionality as a locally cartesian-closed quasi-category. After verifying quasi-categorical notions like initial objects in algebra quasi-categories agree with their counterparts in the internal language of type theory, the desired result should follow (work in progress).

This approach leads us to leave the realm of type-theoretic syntax by working in the semantic domain of quasi-categories. Of course, it is not possible to formalize internally the infinitely many levels of coherence e.g. of the notion of algebra morphisms with their compositional structure. Nor is it needed: since contractibility is expressible internally, we can define notions such as homotopy initial algebras in an ad hoc way by referencing only the first few levels [1]. Only finitely many levels of coherence will be needed at any point. However, several steps in the proof each require the need to explicate an additional layer of coherence, making a direct translation to an internal proof infeasible, even though we conjecture it should be constructively generatable from the quasi-categorical proof.

This furthermore suggests the appropriate level of formalization of the above proof would be in terms of some directed homotopy type theory allowing one to reason about objects behaving like $(\omega, 1)$-categories.

Disclaimer: A version of this abstract has been accepted for presentation at TYPES 2015.

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The geometry of constancy

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May 31, 2015

The papers [1, 2] consider the factorization of constant functions $f : X \to A$ through the propositional truncation $\|X\|$ of their domain, relating it to a number of phenomena, including a generalized version of Hedberg's Theorem that charaterizes those types which are sets. In particular, the above work considers general conditions on X that allow one to get $\|X\| \to X$, thus obtaining the explicit existence of an inhabitant of X from its anonymous existence.

For the purposes of this work, a function $f : X \to A$ is constant if any two of its values are equal:

$$\text{constant}\, f = \Pi(x, y : X).fx = fy.$$

This is not a proposition in general, and one may refer to a point of this type as a modulus of constancy of f. The above papers also give a number of sufficient conditions for the factorization to be possible, and conjecture that the factorization is not possible in general. In particular, if the type A is a set, then the factorization is always possible, and hence one needs to go beyond sets to settle the question.

Mike Shulman (personal communication) exhibited a family of constant functions for which a uniform factorization contradicts the univalence axiom, thus proving the conjecture (unpublished). Not all constant functions $f : X \to A$ factor through $\|X\|$.

Here we look at the problem from a more abstract, geometrical point of view. Moreover, we add a positive factorization result, originally conceived as an attempt to get a negative result, which here is offered as an illustration of the difficulty of the problem solved by Shulman.

Given any type X, we define a universal constant map $X \to S(X)$ by higher-induction. We have the constructors

$$\begin{array}{rcl} \beta & : & X \to S(X), \\ \ell & : & \Pi(x, y : X).\beta(x) = \beta(y). \end{array}$$

When X is the terminal type 1, we have that $S(X)$ is the circle S^1:

$$S(1) = S^1.$$

The universal property of $S(-)$ is an equivalence

$$(S(X) \to A) \simeq \Sigma(f : X \to A).\, \text{constant}\, f,$$

which generalizes the universal property of the circle expressed as

$$(S^1 \to A) \simeq \Sigma(a : A).a = a.$$

To say that a function $f : X \to A$ is constant is equivalent to saying that it factors through $S(X)$. The type $S(X)$ is connected, the unit $\beta : X \to S(X)$ of the universal property of $S(X)$ is a constant surjection, and, because the universal map $X \to \|X\|$ into a proposition is constant, we always have a map $S(X) \to \|X\|$. We have a map $\|X\| \to S(X)$ for all X if and only if all constant functions $f : X \to A$ factor through $\|X\|$. Thus the general factorization problem is equivalent to the question of whether we have a function

$$\Pi(X : U).\|X\| \to S(X).$$

It seemed to us that perhaps the simplest potential counter-example could be

$$X = (s = \text{base}) \text{ for } s : S^1,$$

because then $\|X\| = 1$ as is well known and proved in the HoTT book, and so the question specialized to this particular case amounts to

$$\Pi(s : S^1).S(s = \text{base}).$$

It seemed preposterous to us to always be able to give an element of the type $S(s = \text{base})$ without being able to give an element of the type $(s = \text{base})$ in general. However, this is how things turn out be, and what we will present in the talk.

A consequence of this is that if we are given a point $s : S^1$ and a constant function $f : s = \text{base} \to A$ into a type A, then we can find a point of A which is the constant value of f, even in the absence of the knowledge of a point of the path space $(s = \text{base})$.

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