Eliminating out of Truncations

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General Question

$$\textbf{What is} \quad \|\mathbf{A}\|_{\mathbf{n}} \to \mathbf{B} \quad \textbf{?}$$

I mainly talk about:

$$\textbf{What is} \quad \|\mathbf{A}\| \to \mathbf{B} \quad \textbf{?}$$

(where $\|-\|$ is the propositional truncation, i.e. $n \equiv -1$.)

What is a function $\mathbf{g : \|A\| \to B}$?

A function $f : A \to B$ that cannot look at its input?

$$\mathsf{wconst}_f :\equiv \prod_{a_1, a_2 : A} f(a_1) = f(a_2).$$

Theorem

$$(\|A\| \to B) \quad \simeq \quad \Sigma\,(f : A \to B)\,.\,\mathsf{wconst}_f$$

if B is a 0-type (h-set).

First coherence condition

$$\mathsf{wconst}_f :\equiv \prod_{a_1, a_2 : A} f(a_1) = f(a_2)$$

Coherence condition on $c : \mathsf{wconst}_f$

$$\mathsf{coh}_{f,c} :\equiv \prod_{a^1 a^2 a^3 : A} c(a^1, a^2) \cdot c(a^2, a^3) = c(a^1, a^3).$$

Theorem

$$(\|A\| \to B) \quad \simeq \quad \Sigma (f : A \to B) \,.\, \Sigma (c : \mathsf{wconst}_f) \,.\, \mathsf{coh}_{f,c}$$

if B is a 1-type.

Proof of $(\|A\| \to B) \quad \simeq \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_0 : A$ is given.

B

Proof of $(\|A\| \to B) \simeq \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_0 : A$ is given.

$$\Sigma\,(f_1 : B)\,.$$

$$\mathbf{1}$$

Proof of $(\|A\| \to B) \quad \simeq \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_0 : A$ is given.

$$\Sigma\,(f_1 : B)\,.$$

$$\Sigma\,(f : A \to B)\,.\,\Sigma\,(c_1 : \Pi_{a:A}\ f(a) = f_1)\,.$$

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Proof of $(\|A\| \to B) \quad \simeq \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_\circ : A$ is given.

$$\Sigma\,(f_1 : B)\,.$$

$$\Sigma\,(f : A \to B)\,.\,\Sigma\,(c_1 : \Pi_{a:A}\ f(a) = f_1)\,.$$

$$\Sigma\,(c : \mathsf{wconst}_f)\,.\,\Sigma\left(d_1 : \Pi_{a^1 a^2 : A}\ c(a^1, a^2) \cdot c_1(a^2) = c_1(a^1)\right).$$

$$\Sigma\,(c_2 : f(\mathfrak{a}_\circ) = f_1)\,.\,\Sigma\,(d_3 : c(\mathfrak{a}_\circ, \mathfrak{a}_\circ) \cdot c_1(\mathfrak{a}_\circ) = c_2)\,.$$

$$\mathbf{1}$$

Proof of $(\|A\| \to B) \quad \simeq \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_{\mathfrak{o}} : A$ is given.

$$\Sigma\,(f_1 : B)\,.$$

$$\Sigma\,(f : A \to B)\,.\,\Sigma\,(c_1 : \textstyle\prod_{a:A}\; f(a) = f_1)\,.$$

$$\Sigma\,(c : \mathsf{wconst}_f)\,.\,\Sigma\left(d_1 : \textstyle\prod_{a^1 a^2 : A}\; c(a^1, a^2) \cdot c_1(a^2) = c_1(a^1)\right).$$

$$\Sigma\,(c_2 : f(\mathfrak{a}_{\mathfrak{o}}) = f_1)\,.\,\Sigma\,(d_3 : c(\mathfrak{a}_{\mathfrak{o}}, \mathfrak{a}_{\mathfrak{o}}) \cdot c_1(\mathfrak{a}_{\mathfrak{o}}) = c_2)\,.$$

$$\Sigma\,(d : \mathsf{coh}_{f,c})\,.$$

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Proof of $\quad (\|A\| \to B) \quad \simeq \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

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$\Sigma\,(c_2 : f(\mathfrak{a}_0) = f_1)\,.$ ~~$\Sigma\left(d_3 : c(\mathfrak{a}_0, \mathfrak{a}_0) \cdot c_1(\mathfrak{a}_0) = c_2\right).$~~

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$\Sigma\,(c : \mathsf{wconst}_f)\,.$ ~~$\Sigma\left(d_1 : \prod_{a^1 a^2 : A} c(a^1, a^2) \cdot c_1(a^2) = c_1(a^1)\right).$~~

$\Sigma\,(c_2 : f(\mathfrak{a}_\mathfrak{o}) = f_1)\,.$ ~~$\Sigma\,(d_3 : c(\mathfrak{a}_\mathfrak{o}, \mathfrak{a}_\mathfrak{o}) \cdot c_1(\mathfrak{a}_\mathfrak{o}) = c_2)\,.$~~

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Proof of $(\|A\| \to B) \simeq \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_0 : A$ is given.

$\Sigma\,(f_1 : B)\,.$

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Proof of $(\|A\| \to B) \simeq \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathsf{wconst}_f)\,.\,\mathsf{coh}_{f,c}$

Assume $\mathfrak{a}_0 : A$ is given.

~~$\Sigma\,(f_1 : B)\,.$~~

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$\mathbf{1}$

Assuming $\mathfrak{a}_0 : A$, we have constructed an equivalence

$$g : B \quad \to \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathrm{wconst}_f)\,.\,\mathrm{coh}_{f,c}\,.$$

By examining the steps, we see that the function is

$$g(b) \;\equiv\; (\lambda\_.b\;,\;\; \lambda\_,\_.\mathrm{refl}_b\;,\;\; \lambda\_,\_,\_.\mathrm{refl}_{\mathrm{refl}_b}).$$

It does not depend on $\mathfrak{a}_0$!

$$\begin{aligned} A &\to \mathrm{isequiv}(g) \\ \text{thus} \quad \|A\| &\to \mathrm{isequiv}(g). \end{aligned}$$

Therefore:

$$\begin{aligned} \|A\| \quad &\to \quad (B \;\simeq\; \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathrm{wconst}_f)\,.\,\mathrm{coh}_{f,c}) \\ (\|A\| \to B) \quad &\simeq \quad \Sigma\,(f : A \to B)\,.\,\Sigma\,(c : \mathrm{wconst}_f)\,.\,\mathrm{coh}_{f,c} \end{aligned}$$

This strategy is so frugal that it can be done at any level, with minimalistic assumptions on the theory: we need $\mathbf{1}, \Sigma, \Pi, \mathsf{Id}$ with function extensionality, $\|{-}\|$.

Main result: In a type theory with Reedy ω^{op}-limits (infinite Σ-types), the type $\|A\| \to B$ corresponds to the type of *coherently constant* functions $A \to B$.

Setting: type-theoretic fibration category (Shulman, *Univalence for inverse diagrams and homotopy canonicity*)

Main part of this talk: a very, very rough outline of the proof.

Coherently constant functions are morphisms between semi-simplicial types ($\Delta_+^{\mathrm{op}} \to \mathsf{Type}$)

$$\begin{tikzcd}
A \times A \times A \arrow[r, dashed, "\mathsf{coh}_{f,c}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \begin{array}{c} \Sigma\,(b_1, b_2, b_3 : B)\,. \\ \Sigma\,(p_{12} : b_1 = b_2)\,. \\ \Sigma\,(p_{23} : b_2 = b_3)\,. \\ \Sigma\,(p_{13} : b_1 = b_3)\,. \\ p_{12} \cdot p_{23} = p_{13} \end{array} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
A \times A \arrow[r, dashed, "c : \mathsf{wconst}_f"] \arrow[d, shift left=1] \arrow[d, shift right=1] & \Sigma\,(b_1, b_2 : B)\,.\, b_1 = b_2 \arrow[d, shift left=1] \arrow[d, shift right=1] \\
A \arrow[r, dashed, "f"] & B \\
\mathcal{T}A :\ \Delta_+^{\mathrm{op}} \to \mathsf{Type} & \mathcal{E}B :\ \Delta_+^{\mathrm{op}} \to \mathsf{Type}
\end{tikzcd}$$

[0]-coskeleton of A

Fibrant replacement of B

On the Equality Semi-Simplicial Type $\mathcal{E}B$

$\mathcal{E}B_{[\mathbf{n}]}$ is the type of *n*-dimensional tetrahedra, built of the identity type (defined as a Shulman-kind diagram over the inverse category Δ_+^{op}). We can also define the type of *horns*.

Important Kan-filling lemma: **The projection from full tetrahedra to the type of (k-)horns is an equivalence.**

(Side remark: This is a strong “Kan filling” property and gives a “simplicial” version of Lumsdaine’s / van den Berg-Garner’s “globular” result that types are weak ω-groupoids.)

Nat. trans. between $\widehat{\mathcal{T}}A$ and $\widehat{\mathcal{E}}B$ (extended index cat. $\widehat{\Delta}_+^{\mathrm{op}}$)

$$\begin{tikzcd}
d_1 : \prod_{a^1 a^2 : A} \; c(a^1, a^2) \cdot c_1(a^2) = c_1(a^1) \arrow[ddd] \arrow[ddddr] & & & \\
d : \mathrm{coh}_{f,c} \arrow[dd] & & d_2 : \prod_{a:A} \; c(\mathfrak{a}_0, a) \cdot c_1(a) = c_2 \arrow[ddll] \arrow[dddl] \arrow[ddd] & \\
 & & & d_3 : c(\mathfrak{a}_0, \mathfrak{a}_0) \cdot c_1(\mathfrak{a}_0) = c_2 \arrow[dlll] \arrow[ddll] \arrow[ddl] \\
c : \mathrm{wconst}_f \arrow[dd] & & & \\
 & c_1 : \prod_{a:A} \; f(a) = f_1 \arrow[dl] \arrow[d] & c_2 : f(\mathfrak{a}_0) = f_1 \arrow[dll] \arrow[dl] & \\
f : A \to B & f_1 : B & &
\end{tikzcd}$$

Kan-filling lemma ⇒ . . . extensive calculation . . . ⇒ Any two Σ-components connected by a “diagonal arrow” form a contractible pair!

Nat. trans. between $\widehat{\mathcal{T}}A$ and $\widehat{\mathcal{E}}B$ (extended index cat. $\widehat{\Delta}_+^{\mathrm{op}}$)

$$\begin{tikzcd}
 & d_1 : \prod_{a^1 a^2 : A} c(a^1, a^2) \cdot c_1(a^2) = c_1(a^1) \arrow[ddddl] \arrow[ddddd] & & \\
d : \mathrm{coh}_{f,c} \arrow[ddd] & & & \\
 & & d_2 : \prod_{a:A} c(\mathfrak{a}_0, a) \cdot c_1(a) = c_2 \arrow[ddll] \arrow[dddl] \arrow[ddd] & \\
 & & & d_3 : c(\mathfrak{a}_0, \mathfrak{a}_0) \cdot c_1(\mathfrak{a}_0) = c_2 \arrow[dlll] \arrow[ddll] \arrow[ddl] \\
c : \mathrm{wconst}_f \arrow[dd] & & & \\
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\end{tikzcd}$$

Kan-filling lemma ⇒ . . . extensive calculation . . . ⇒ Any two Σ-components connected by a "diagonal arrow" form a contractible pair!

Rest as in the special case:

- Assuming $\mathfrak{a}_0 : A$, we have shown that the can. map

  $$B \;\to\; \text{nat. trans. from } \mathcal{T}A \text{ to } \mathcal{E}B$$

  is an equivalence.
- This map is independent of $\mathfrak{a}_0$.
- Thus, $\|A\|$ implies that this map is an equivalence.
- Therefore:

Theorem

$$(\|A\| \to B) \quad \simeq \quad \text{nat. trans. from } \mathcal{T}A \text{ to } \mathcal{E}B$$

in any theory with $\mathbf{1}, \Sigma, \Pi, \mathsf{Id}$, fun.ext., $\|-\|$,
Reedy ω^{op}-limits.

If you don't like Reedy ω-limits, you still get all the cases where B is n-truncated.

Higher Truncations

What is $\|A\|_n \to B$?

Conjecture: Natural Transformations from the
$[n+1]$-coskeleton of $\mathcal{E}A$ to $\mathcal{E}B$.

This talk: Case $n \equiv -1$.

Paolo Capriotti, N.K., Andrea Vezzosi: Proof for the case
that B is $(n+1)$-truncated (to appear at CSL'15).

Caveat, wild speculation following.

Case $n \equiv 0$ can be used to solve the open problem
"**univalent type theory eats itself**"
with n univalent universes, but without HITs; trick:
interpret $\mathcal{U}_i$ as universe of i-types.