Classifying types (jww Ulrik Buchholtz)

and

The join construction

Egbert Rijke

Carnegie Mellon University

erijke@andrew.cmu.edu

June 25th 2016, Porto

Part I: Classifying types (jww Ulrik Buchholtz)

Overview

- Definition of $\mathbb{R}\mathrm{P}^n$.
- Definition of $\mathbb{C}\mathrm{P}^n$.
- Principal H-spaces.
- The general projective space construction.
- The principal H-space structure of loop spaces.
- Principal H-spaces are pointed connected types.

$$\mathbb{R}\mathsf{P} : \mathbb{N}_{-1} \to \mathsf{U}$$

$$\mathsf{cov}_2 : \prod_{(n:\mathbb{N}_{-1})} \mathbb{R}\mathsf{P}^n \to \mathsf{U}_{\mathbf{2}}$$

$$\begin{tikzcd}
\sum_{(x:\mathbb{R}\mathsf{P}^n)} \mathsf{cov}_2^n(x) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & \mathbf{1} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{R}\mathsf{P}^n \arrow[r] \arrow[drr, bend right, "\mathsf{cov}_2^n"'] & \mathbb{R}\mathsf{P}^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\mathsf{cov}_2^{n+1}"'] & \\
 & & \mathsf{U}_{\mathbf{2}}
\end{tikzcd}$$

- For each $n : \mathbb{N}$ there is an equivalence

$$\mathbb{S}^n \simeq \sum_{(x:\mathbb{R}\mathrm{P}^n)} \mathrm{cov}_2^n(x).$$

  In other words, $\mathbb{R}\mathrm{P}^{n+1}$ is obtained by attaching an $(n+1)$-cell to $\mathbb{R}\mathrm{P}^n$.
- We obtain the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\mathbb{R}\mathrm{P}^n) \longrightarrow \pi_k(\mathbf{2}) \longrightarrow \pi_k(\mathbb{S}^n) \longrightarrow \pi_k(\mathbb{R}\mathrm{P}^n) \longrightarrow \cdots$$

  Since $\pi_k(\mathbf{2}) = 0$ for $k \geq 1$, we get the isomorphisms

$$\pi_k(\mathbb{S}^n) = \pi_k(\mathbb{R}\mathrm{P}^n)$$

  for $k \geq 2$.
- The map

$$\mathrm{cov}_2^\infty : \mathbb{R}\mathrm{P}^\infty \to \mathrm{U}_\mathbf{2}$$

  is an equivalence.

$$\mathcal{O}_{\mathbb{S}^1} : \mathsf{U}_{\mathbb{S}^1} \to \mathsf{U}$$

such that $\sum_{(A:\mathsf{U}_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A) \times A$ is contractible

$$\begin{aligned} \mathbb{C}\mathsf{P} &: \mathbb{N}_{-1} \to \mathsf{U} \\ \mathsf{cov}_{\mathbb{S}^1} &: \prod_{(n:\mathbb{N}_{-1})} \mathbb{C}\mathsf{P}^n \to \mathsf{U}_{\mathbb{S}^1} \\ \mathsf{orient}_{\mathbb{S}^1} &: \prod_{(n:\mathbb{N}_{-1})} \prod_{(x:\mathbb{C}\mathsf{P}^n)} \mathcal{O}_{\mathbb{S}^1}(\mathsf{cov}^n_{\mathbb{S}^1}(x)) \end{aligned}$$

$$\begin{tikzcd}
\sum_{(x:\mathbb{C}\mathsf{P}^n)} \mathsf{cov}^n_{\mathbb{S}^1}(x) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & \mathbf{1} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{C}\mathsf{P}^n \arrow[r] \arrow[drr, bend right, "\langle \mathsf{cov}^n_{\mathbb{S}^1}, \mathsf{orient}^n_{\mathbb{S}^1} \rangle"'] & \mathbb{C}\mathsf{P}^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\mathsf{cov}^{n+1}_{\mathbb{S}^1}"'] & \\
& & \sum_{(A:\mathsf{U}_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A)
\end{tikzcd}$$

- For each $n : \mathbb{N}$ there is an equivalence

$$\mathbb{S}^{2n+1} \simeq \sum_{(x:\mathbb{C}\mathsf{P}^n)} \mathsf{cov}^n_{\mathbb{S}^1}(x).$$

  In other words, $\mathbb{C}\mathsf{P}^{n+1}$ is obtained by attaching an $(2n+2)$-cell to $\mathbb{C}\mathsf{P}^n$.
- We obtain the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\mathbb{C}\mathsf{P}^n) \longrightarrow \pi_k(\mathbb{S}^1) \longrightarrow \pi_k(\mathbb{S}^{2n+1}) \longrightarrow \pi_k(\mathbb{C}\mathsf{P}^n) \longrightarrow \cdots$$

  Since $\pi_k(\mathbb{S}^1) = 0$ for $k \geq 2$, we get the isomorphisms

$$\pi_k(\mathbb{S}^{2n+1}) = \pi_k(\mathbb{C}\mathsf{P}^n)$$

  for $k \geq 3$.
- The map

$$\mathsf{cov}^\infty_{\mathbb{S}^1} : \mathbb{C}\mathsf{P}^\infty \to \sum_{(A:\mathsf{U}_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A)$$

  is an equivalence.

Definition

A principal H-space structure on a type X with base point 1_X, consists of

1. a type family $\mathcal{O}_X : \mathsf{U}_X \to \mathsf{U}$ of orientations,
2. a canonical orientation $o_X : \mathcal{O}_X(X)$,

such that the type

$$\sum_{(A:U_X)} \mathcal{O}_X(A) \times A$$

is contractible.

The classifying type of a principal H-space X is defined to be

$$\mathbf{B}X :\equiv \sum_{(A:\mathsf{U}_X)} \mathcal{O}_X(A).$$

This is a pointed connected type with loop space X.

The general projective space construction

$$\begin{aligned}
\mathsf{P}(X) &: \mathbb{N}_{-1} \to \mathsf{U} \\
\mathsf{cov}_X &: \textstyle\prod_{(n:\mathbb{N}_{-1})} \mathsf{P}(X)^n \to \mathsf{U}_X \\
\mathsf{orient}_X &: \textstyle\prod_{(n:\mathbb{N}_{-1})} \prod_{(x:\mathsf{P}(X)^n)} \mathcal{O}_X(\mathsf{cov}_X^n(x))
\end{aligned}$$

$$\begin{tikzcd}
\sum_{(x:\mathsf{P}(X)^n)} \mathsf{cov}_X^n(x) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & \mathbf{1} \arrow[d] \arrow[ddr, bend left] & \\
\mathsf{P}(X)^n \arrow[r] \arrow[drr, bend right, "\langle \mathsf{cov}_X^n, \mathsf{orient}_X^n \rangle"'] & \mathsf{P}(X)^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\mathsf{cov}_X^{n+1}"'] & \\
 & & \sum_{(A:\mathsf{U}_X)} \mathcal{O}_X(A)
\end{tikzcd}$$

- For each $n : \mathbb{N}$ there is an equivalence

$$X^{*(n+1)} \simeq \sum_{(x:\mathsf{P}(X)^n)} \mathsf{cov}_X^n(x).$$

- We obtain the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\mathsf{P}(X)^n) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X^{*(n+1)}) \longrightarrow \pi_k(\mathsf{P}(X)^n) \longrightarrow \cdots$$

- The map

$$\mathsf{cov}_X^\infty : \mathsf{P}(X)^\infty \to \mathbf{B}X$$

  is an equivalence.

Lemma

Any loop space can be given the structure of a principal H-space.

Proof (Definition of the type of orientations).

Define $P : X_{x_0} \to \mathrm{Type}$ by $P(x) :\equiv (x = x_0)$. Then we have

$$\begin{tikzcd}
\Omega(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[r] \arrow[d, "x_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U}_X \arrow[d] \\
\mathbf{1} \arrow[r, "x_0"'] & X_{x_0} \arrow[r, "P"'] & \mathrm{U}_X
\end{tikzcd}$$

Now take

$$\mathcal{O}_X(A) :\equiv \mathrm{fib}_P(A)$$
$$o_X(\Omega(X)) :\equiv \langle x_0, \mathrm{refl}_{\Omega(X)} \rangle$$

$\square$

Proof of the contractibility property.

We have a commuting triangle

$$\begin{tikzcd} X_{x_0} \arrow[rr, "\simeq"] \arrow[dr, "P"'] & & \sum_{(A:U_X)} \mathrm{fib}_P(A) \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{U}_X & \end{tikzcd}$$

Therefore, the following are equivalent

- $\sum_{(A:U_X)} \mathcal{O}_X(A) \times A$ is contracitble
- $\sum_{(x:X_{x_0})} P(x)$ is contractible

The latter is obvious. $\square$

Observation: By the projective space construction, $\mathbf{B}X$ is equivalent to a small type, for any principal H-space X.

Theorem

The map $\mathbf{B}$ *from principal H-spaces to the type of (small) pointed connected types is an equivalence.*

Theorem

The type $\mathbf{B}X$ *classifies the oriented* X*-bundles over types:*
For any $f : A \to B$ *with the property that* $\|\mathrm{fib}_f(y) = X\|$ *for all* $y : B$*, the following are equivalent:*

- *the type of all maps* $P : B \to \mathbf{B}X$ *for which the square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \mathbf{1} \arrow[d] \\ B \arrow[r, "P"'] & \mathbf{B}X \end{tikzcd}$$

  is a pullback square
- *the type* $\prod_{(y:B)} \mathcal{O}_X(\mathrm{fib}_f(y))$ *of* X*-orientations of* f

Part II: The join construction

Overview

- The join construction
- The modified join construction
- The construction of the modality of $\bigcirc$-separated types
- Principal equivalence relations
- The type of principal equivalence relations on X is equivalent to the type of surjective maps out of X.

$$\begin{tikzcd}
A \times_X B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\mathsf{inr}"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "\mathsf{inl}"'] \arrow[drr, bend right, "f"'] & A *_X B \arrow[dr, dotted, "f*g"] & \\
& & X
\end{tikzcd}$$

Theorem

If f is M-connected and g is N-connected for two types M and N, then $f * g$ is $(M * N)$-connected.

Corollary

If f is m-connected and g is n-connected for $m, n : \mathbb{N}$, then $f * g$ is $(m + n + 2)$-connected.

Definition

For any $f : A \to X$, we define a sequence

$$\begin{tikzcd}
A_0 \arrow[r, "i_0"] \arrow[dr, "f^{*0}"'] & A_1 \arrow[r, "i_1"] \arrow[d, "f^{*1}"] & A_2 \arrow[r, "i_2"] \arrow[dl, "f^{*2}"] & \cdots \arrow[dll, "f^{*3}"] \\
& X & &
\end{tikzcd}$$

The function f^{*n} is called the n-th join-power of f.

Construction.

We take $A_0 :\equiv \mathbf{0}$, with the unique map into X. Then we define $A_{n+1} :\equiv A_n *_X A$, and $f^{*(n+1)} :\equiv f^{*n} * f$. $\square$

For example:

- In the real projective case:
  - $\mathbb{R}\mathsf{P}^{n+1} = \mathbb{R}\mathsf{P}^n *_{\mathbb{R}\mathsf{P}^\infty} \mathbf{1}$, and
  - $\mathrm{cov}_2^n = (\mathrm{cov}_2^0)^{*(n+1)}$
- In the complex projective case:
  - $\mathbb{C}\mathsf{P}^{n+1} = \mathbb{C}\mathsf{P}^n *_{\mathbb{C}\mathsf{P}^\infty} \mathbf{1}$, and
  - $\mathrm{cov}_{\mathbb{S}^1}^n = (\mathrm{cov}_{\mathbb{S}^1}^0)^{*(n+1)}$
- In the general projective case:
  - $\mathsf{P}(X)^{n+1} = \mathsf{P}(X)^n *_{\mathbf{B}X} \mathbf{1}$, and
  - $\mathrm{cov}_X^n = (\mathrm{cov}_X^0)^{*(n+1)}$

Theorem

The sequential colimit $f^{*\infty}$ is an embedding, and has the universal property of the image inclusion.

Corollary

The sequential colimit of the type sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{\mathsf{inr}} A * A \xrightarrow{\mathsf{inr}} A * (A * A) \xrightarrow{\mathsf{inr}} \cdots$$

has the universal property of the propositional truncation.

Definition

A type X, which may itself be large, is said to be locally small if for all $x, y : X$, there is a type $x =' y : \mathsf{U}$ and an equivalence of type

$$(x = y) \simeq (x =' y).$$

Examples of locally small types include:

- all types in U,
- the universe U,
- mere propositions of any size,
- for any $A : \mathsf{U}$ and any locally small type X, the type $A \to X$.

$$
\begin{tikzcd}
A \times'_X B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\mathsf{inr}"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "\mathsf{inl}"'] \arrow[drr, bend right, "f"'] & A *'_X B \arrow[dr, dotted, "f *' g"'] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & X
\end{tikzcd}
$$

Theorem

Assumptions:

- U *is a univalent universe closed under graph quotients,*
- *let* X *be a locally small type,*
- *let* $f : A \to X$ *with* $A : \mathsf{U}$,

Then we can construct a type $\mathrm{im}'(f) : \mathsf{U}$, *a surjective map* $q'_f : A \to \mathrm{im}'(f)$, *and an embedding* $i'_f : \mathrm{im}'(f) \to X$ *such that the triangle*

$$\begin{tikzcd} A \arrow[r, "q'_f"] \arrow[dr, "f"'] & \mathrm{im}'(f) \arrow[d, "i'_f"] \\ & X \end{tikzcd}$$

commutes, with the universal property of the image inclusion of f.

Applications of the modified join construction

Corollary

For any $\mathsf{Prop}_{\mathsf{U}}$*-valued equivalence relation* $R : A \to A \to \mathsf{Prop}_{\mathsf{U}}$ *over a type* $A : \mathsf{U}$*, the type* $A/R :\equiv \mathsf{im}'(R) : \mathsf{U}$ *has the universal property of the quotient.*

Corollary

The Rezk completion of any small precategory A *can be constructed in any univalent universe that is closed under graph quotients, and the Rezk completion of any small precategory is again a small category.*

Definition

A type $A : \mathrm{U}$ is said to be $\bigcirc$-separated if the type $a = b$ is $\bigcirc$-modal, for any $a, b : A$.

- Any $(n+1)$-truncated type is $\|-\|_n$-separated.

Definition

Let $\bigcirc$ be a modality and let $A : \mathrm{U}$. We define the reflexive relation $I_\bigcirc(A) : A \to A \to \mathrm{U}$ by

$$I_\bigcirc(A)(a, b) :\equiv \bigcirc(a = b).$$

Theorem

For any type A and any $a : A$, the type

$$\sum\nolimits_{(P:\mathrm{im}'_t(I_{\bigcirc}(A)))} P(a)$$

is contractible.

It follows by the encode-decode method that

$$\bigcirc(a = b) \simeq (I_{\bigcirc}(A)(a) = I_{\bigcirc}(A)(b)).$$

for any $a, b : A$.

Definition

Let $\bigcirc$ be any modality on U, and let $A : \mathrm{U}$. We define

$$\bigcirc^{+}(A) :\equiv \mathrm{im}'(I_{\bigcirc}(A)).$$

Furthermore, we define the modal unit $\eta_{\bigcirc^{+}}$ by

$$\eta_{\bigcirc^{+}}(A) :\equiv q'_{I_{\bigcirc}(A)}.$$

Theorem

The operation $\bigcirc^{+}$ defined in this way is a modality.

Corollary

The n-truncations are definable operations in a univalent universe closed under graph quotients.

Definition

A principal equivalence relation on a type A consists of

- A binary relation $R : A \to (A \to \mathrm{U})$ with a proof $\rho : \prod_{(a:A)} R(a,a)$ of reflexivity,
- A type family

$$\mathcal{O}_R : \mathrm{im}'(R) \to \mathrm{U}$$

of R-orientations on the predicates $P : A \to \mathrm{U}$ in the image of R, with a canonical R-orientation

$$o_R : \prod_{(a:A)} \mathcal{O}_R(R(a)),$$

such that the type

$$\sum_{(P:\mathrm{im}'(R))} \mathcal{O}_R(P) \times P(a)$$

is contractible for every $a : A$.

- A principal equivalence relation on $\mathbf{1}$ is the same thing as a principal H-space.

Definition

Given a principal equivalence relation R on A, we define the quotient

$$A/R :\equiv \sum_{(P:\mathrm{im}'(R))} \mathcal{O}_R(P).$$

We define the quotient map $q_R : A \to A/R$ by

$$q_R(a) :\equiv \langle q'_R(a), o_R(a) \rangle.$$

We obtain a map $\mathcal{Q}_A$ from the type of principal equivalence relations on A to the type of surjective maps out of A.

Theorem

For each type A, the map $\mathcal{Q}_A$ is an equivalence.