Fibred Fibration Categories

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Fibred type-theoretic fibration categories

A “good” notion of fibred category between categorical models of dependent type theory.

- Construct total type-theoretic structure from fiberwise one
- Change of base

Application

Categorical description of “logical relations” [Hermida, 1993] on HoTT.

Theorem

\[ t : \Pi_{A:U} \Pi_{x:A} x = x \rightarrow x = x \text{ is homotopic to } \lambda(p : x = x).p^n \text{ for some } n \in \mathbb{Z}. \]
Type-theoretic fibration categories [Shulman, 2015] are sound and complete categorical models of Martin-Löf’s intentional theory.

- A category $\mathcal{C}$ equipped with specific morphisms called *fibrations* corresponding to type families.
- Path induction is modeled by the *lifting* property.

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\text{refl} & \downarrow & \downarrow \\
\text{Id}_A & \longrightarrow & \text{Id}_A
\end{array}
\]

- A morphism like refl is called an *acyclic cofibration*.
Fibred type-theoretic fibration categories

A fibred type-theoretic fibration category is a fibred category \( p : \mathbb{E} \rightarrow \mathbb{B} \) such that:

1. \( \mathbb{E} \) and \( \mathbb{B} \) are type-theoretic fibration categories and \( p \) preserves all structures of type-theoretic fibration category.

2. A fibration in \( \mathbb{E} \) factors as a vertical fibration followed by a horizontal fibration.

\[
\begin{array}{c}
\mathbb{E} \\
p \downarrow \\
\mathbb{B}
\end{array} \quad \begin{array}{c}
X \\
\downarrow \quad \downarrow \\
\mathbb{E}_{\mathbb{B}} \quad u^* \mathbb{Y} \rightarrow \mathbb{Y}
\end{array} \quad \begin{array}{c}
\mathbb{Y} \\
\alpha \quad \downarrow u \quad \beta
\end{array}
\]

3. Every Cartesian morphism above an acyclic cofibration is an acyclic cofibration.
Proposition

A fibred category \( p : \mathcal{E} \to \mathcal{B} \) is a fibred type-theoretic fibration category if and only if:

1. \( \mathcal{B} \) and all fibers \( \mathcal{E}_\alpha \) are type-theoretic fibration categories.

2. For any morphism \( u : \alpha \to \beta \) in \( \mathcal{B} \), \( u^* : \mathcal{E}_\beta \to \mathcal{E}_\alpha \) is a strong fibration functor.

3. For every acyclic cofibration \( u : \alpha \xrightarrow{\sim} \beta \) and fibration \( f : Y \to X \) in \( \mathcal{E}_\beta \), 
\( u^* : \mathcal{E}_\beta/X(X,Y) \to \mathcal{E}_\alpha/u^*X(u^*X,u^*Y) \) is surjective.

4. For every fibration \( u : \alpha \to \beta \) in \( \mathcal{B} \), 
\( u^* : \mathcal{E}_\beta \to \mathcal{E}_\alpha \) has a right adjoint \( u_* \) satisfying the “Beck-Chevalley condition”.
Proof of the Proposition, syntactically

A fully categorical proof is in arXiv:1602.08206. I give a syntactic description. A fibred category $p : \mathcal{E} \to \mathcal{B}$ models a type theory with two sorts Kind and Type.

<table>
<thead>
<tr>
<th>Category theory</th>
<th>Type theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \in \mathcal{B}$</td>
<td>$\alpha : \text{Kind}$</td>
</tr>
<tr>
<td>$X \in \mathcal{E}_\alpha$</td>
<td>$X : \alpha \to \text{Type}$</td>
</tr>
</tbody>
</table>

The Proposition says the pairs of $(\alpha : \text{Kind}, X : \alpha \to \text{Type})$ form a new type theory.
Proof of the Proposition, syntactically

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Definition</th>
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<tbody>
<tr>
<td>type</td>
<td>$(\alpha : \text{Kind}, X : \alpha \rightarrow \text{Type})$</td>
</tr>
<tr>
<td>element</td>
<td>$(a : \alpha, x : X(a))$</td>
</tr>
<tr>
<td>family</td>
<td>$(\beta : \alpha \rightarrow \text{Kind}, Y : \Pi a : \alpha \beta(a) \rightarrow X(a) \rightarrow \text{Type})$</td>
</tr>
<tr>
<td>section</td>
<td>$(u : \Pi a : \alpha \beta(a), f : \Pi a : \alpha \Pi x : X(a) Y(a, u(a), x))$</td>
</tr>
<tr>
<td>pair</td>
<td>$((a, b) : \Sigma a : \alpha \beta(a), (x, y) : \Sigma x : X(a) Y(a, b, x))$</td>
</tr>
<tr>
<td>identity type</td>
<td>?</td>
</tr>
</tbody>
</table>
Proof of the Proposition, syntactically

We need the path induction on an identity kind w.r.t. any type family over the identity kind.

\[ \alpha : \text{Kind} \]
\[ X : \Pi_{a,a':\alpha} a = a' \rightarrow \text{Type} \]
\[ x : \Pi_{a:\alpha} X(a, a, \text{refl}_a) \]
\[ \text{ind}_{=\alpha}(X, x) : \Pi_{a,a':\alpha} \Pi_{p:a=a'} X(a, a', p) \]
\[ \text{ind}_{=\alpha}(X, x, a, a, \text{refl}_a) \equiv x \]

In particular, for \( \alpha : \text{Kind} \), \( X : \alpha \rightarrow \text{Type} \) and \( p : a =_{\alpha} a' \), we have the transport along \( p \)
\[ p_* : X(a) \rightarrow X(a') \].
Proof of the Proposition, syntactically

The identity type of \((\alpha : \text{Kind}, X : \alpha \rightarrow \text{Type})\) is the pair of

\[\begin{align*}
= &: \alpha \rightarrow \alpha \rightarrow \text{Kind} \\
\lambda a a' p x x' . p_\ast x &= x' : \prod_{a, a' : \alpha} \prod_{p : a = a'} X(a) \rightarrow X(a') \rightarrow \text{Type}.
\end{align*}\]

It is the type of “path over path” but in different sorts.
Universes in a fibred setting

Let $\mathcal{U} : \text{Kind}$ and $\mathcal{V} : \text{Type}$ be universes of kinds and types. Then $(\mathcal{U}, \lambda (\alpha : \mathcal{U}). \alpha \to \mathcal{V})$ is a universe in the new type theory if, for any $\alpha : \mathcal{U}$ and $X : \alpha \to \mathcal{V}$, $\prod_{\alpha : \mathcal{U}} X(\alpha) : \mathcal{V}$. Its elements are $(\alpha : \mathcal{U}, X : \alpha \to \mathcal{V})$. 
Equivalences in a fibred setting

For \((u : \alpha \rightarrow \beta, f : \Pi_{a:\alpha} X(a) \rightarrow Y(u(a)))\), An element of is-equiv\((u, f)\) is \((v : \text{homotopy inverse of } u, g : \text{homotopy inverse of } f \text{ above } v)\).

Lemma
Suppose the function extensionality holds. Then

\[\text{is-equiv}(u, f) \simeq (\text{is-equiv}(u), \lambda _. \Pi_{a:\alpha} \text{is-equiv}(f_a))\]

for all \((u : \alpha \rightarrow \beta, f : \Pi_{a:\alpha} X(a) \rightarrow Y(u(a)))\) in the new type theory.
Univalence in a fibred setting

A universe $U$ in a type theory is \textit{univalent} if the canonical map
$$\lambda(A : U).(A, A, \text{id}_A) : U \rightarrow \Sigma_{A,A' : U} A \simeq A'$$
is an equivalence.

The new universe $(U, \lambda(\alpha : U).\alpha \rightarrow V)$ is univalent if
$$\lambda(\alpha : U).(\alpha, \alpha, \text{id}_\alpha)$$is an equivalence and for all $\alpha : U$, $\lambda X.(X, X, \lambda(a : \alpha).\text{id}_{X(a)}) : (\alpha \rightarrow V) \rightarrow \Sigma_X, Y : \alpha \rightarrow V \Pi_a : \alpha X(a) \simeq Y(a)$ is an equivalence. This holds if $U$ and $V$ are univalent and the function extensionality holds.
Arrow categories

Let $\mathcal{C}$ be a type-theoretic fibration category and write $(\mathcal{C} \to)_f \subset \mathcal{C} \to$ for the full subcategory of all the fibrations. Then $\text{cod} : (\mathcal{C} \to)_f \to \mathcal{C}$ is a fibred type-theoretic fibration category. In this case $\text{Type} \equiv \text{Kind}$.

If $\mathcal{C}$ has a univalent universe, so does $(\mathcal{C} \to)_f$.

Originally, Shulman proved $(\mathcal{C} \to)_f$ is a type-theoretic fibration category [Shulman, 2015], and I give a fibred categorical description.
Change of base

Let $p : \mathcal{E} \to \mathcal{B}$ be a fibred type-theoretic fibration category and $F : \mathcal{A} \to \mathcal{B}$ be a functor preserving fibrations, pullbacks of fibrations and acyclic cofibrations. Then the change of base or pullback

\[
\begin{array}{ccc}
F^*\mathcal{E} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow p \\
\mathcal{A} & \longrightarrow & \mathcal{B}
\end{array}
\]

is a fibred type-theoretic fibration category.
Relational model

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibred type-theoretic fibration category. We have the change of base

$$\mathcal{R}el(p) \to \mathbb{E}$$

$$\downarrow \quad \downarrow p$$

$$\mathbb{B} \xrightarrow{\lambda \alpha. \alpha \times \alpha} \mathbb{B}.$$  

$\mathcal{R}el(p)$ is the category of binary families $(\alpha : \text{Kind}, R : \alpha \to \alpha \to \text{Type})$. 
Application

Let $t : \prod_{A:U} \prod_{x:A} x = x \to x = x$.

Theorem

- $t$ is “natural”: for any $f : A \to B$ in $U$, $\text{ap}_f \circ t \sim t \circ \text{ap}_f$.

$$
\begin{array}{c}
x = x \xrightarrow{t} x = x \\
\text{ap}_f \downarrow \quad \text{ap}_f \\
fx = fx \xrightarrow{t} fx = fx
\end{array}
$$

- If the type theory has $S^1$, then for some $n \in \mathbb{Z}$, $tp = p^n$ for all $p : x = x$. 
Let $\mathbb{C}$ be the syntactic category of Martin-Löf type theory with a univalent universe $\mathbb{U}$. Then 

$$(\mathbb{U}, \lambda(A, B : \mathbb{U}).A \rightarrow B \rightarrow \mathbb{U})$$

is a univalent universe in $\mathcal{Rel}(\text{cod})$. We have a strong fibration functor

![Diagram](image)

In particular, for every closed term $t : A$, we have $R_A : A \rightarrow A \rightarrow \text{Type}$ and $R_t : R_A(t, t)$. 
Application

Let \( t : \Pi_{A:U} \Pi_{x:A} x = x \rightarrow x = x \). Then \( R_t : \Pi_{A,B:U,W:A \rightarrow B \rightarrow U} \Pi_{x:A,y:B,v:W(x,y)} \Pi_{p:x=x,q:y=y} (p, q)_* v = v \rightarrow (tp, tq)_* v = v \).

\[
\begin{array}{c}
\frac{X}{p} \\
\frac{X}{v} \\
\frac{y}{q} \\
\frac{y}{v} \\
\frac{X}{tp} \\
\frac{X}{v} \\
\frac{y}{tq} \\
\frac{y}{v}
\end{array}
\]
Application

Given \( f : A \to B \) in \( \mathcal{U} \), let \( W(x, y) \equiv fx = y \). Then \( R_t \) looks like

\[
\begin{align*}
fx & \xrightarrow{\text{ap}_fp} fx \\
\vdash & \quad \vdash \\
y & \equiv q \quad y \\
\end{align*}
\]

\[
\begin{align*}
fx & \xrightarrow{\text{ap}_f(tp)} fx \\
\vdash & \quad \vdash \\
y & \equiv tq \quad y.
\end{align*}
\]

Let \( y \equiv fx \), \( q \equiv \text{ap}_fp \) and \( v \equiv \text{refl}_{fx} \) and apply the function to \( \text{refl}_{ap_f p} \). Then \( \text{ap}_f(tp) = t(ap_f p) \) for all \( x : A \) and \( p : x = x \).
Application

Suppose the type theory has $S^1 : U$ with $b : S^1$ and $l : b = b$. Then $(S^1, =_{S^1})$ is a unit circle in $\mathcal{R}el(\text{cod})$ and we still have the functor $R : \mathcal{C} \to \mathcal{R}el(\text{cod})$. Let $y : B$ and $q : y = y$ which corresponds to $f : S^1 \to B$. $t(l) = l^n$ for some $n \in \mathbb{Z}$ and

\[
\begin{align*}
tq &= t(ap_f(l)) \\
&= ap_f(t(l)) \\
&= ap_f(l^n) \\
&= (ap_f(l))^n \\
&= q^n
\end{align*}
\]
*Fibrations, Logical Predicates and Indeterminates.*

Univalence for inverse diagrams and homotopy canonicity.
*Mathematical Structures in Computer Science,*
25(05):1203–1277.