Fibred Fibration Categories

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Fibred type-theoretic fibration categories

A “good” notion of fibred category between categorical models of dependent type theory.

- Construct total type-theoretic structure from fiberwise one
- Change of base

Application

Categorical description of “logical relations” [Hermida, 1993] on HoTT.

Theorem

$t : \Pi_{A:U} \Pi_{x:A} x = x \; ! \; x = x$ is homotopic to $\lambda(p : x = x).p^n$ for some $n \; 2 \; \mathbb{Z}$.

Type-theoretic fibration categories

Type-theoretic fibration categories [Shulman, 2015] are sound and complete categorical models of Martin-Löf's intentional theory.

- A category $\mathbb{C}$ equipped with specific morphisms called *fibrations* corresponding to type families.
- Path induction is modeled by the *lifting property*.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{refl}"'] & X \arrow[d, two heads] \\ \mathrm{Id}_A \arrow[r, equal] \arrow[ur, dotted, "J"] & \mathrm{Id}_A \end{tikzcd}$$

  - A morphism like refl is called an *acyclic cofibration*.

Fibred type-theoretic fibration categories

A fibred type-theoretic fibration category is a fibred category $p : \mathbb{E} \to \mathbb{B}$ such that:

1. $\mathbb{E}$ and $\mathbb{B}$ are type-theoretic fibration categories and p preserves all structures of type-theoretic fibration category.
2. A fibration in $\mathbb{E}$ factors as a *vertical* fibration followed by a *horizontal* fibration.

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "p"'] & X \arrow[d, dotted, two heads] \arrow[dr, two heads] & \\
\mathbb{B} & u\,Y \arrow[r, dotted, two heads] & Y \\
 & \alpha \arrow[r, two heads, "u"'] & \beta
\end{tikzcd}
$$

3. Every Cartesian morphism above an acyclic cofibration is an acyclic cofibration.

Proposition

A fibred category $p : \mathbb{E} \to \mathbb{B}$ is a fibred type-theoretic fibration category if and only if:

1. $\mathbb{B}$ and all fibers $\mathbb{E}_\alpha$ are type-theoretic fibration categories.
2. For any morphism $u : \alpha \to \beta$ in $\mathbb{B}$, $u : \mathbb{E}_\beta \to \mathbb{E}_\alpha$ is a strong fibration functor.
3. For every acyclic cofibration $u : \alpha \rightarrowtail \beta$ and fibration $f : Y \twoheadrightarrow X$ in $\mathbb{E}_\beta$, $u : \mathbb{E}_\beta/X(X, Y) \to \mathbb{E}_\alpha/u X(u X, u Y)$ is surjective.
4. For every fibration $u : \alpha \twoheadrightarrow \beta$ in $\mathbb{B}$, $u : \mathbb{E}_\beta \to \mathbb{E}_\alpha$ has a right adjoint u satisfying the “Beck-Chevalley condition”.

Proof of the Proposition, syntactically

A fully categorical proof is in arXiv:1602.08206. I give a syntactic description.

A fibred category $p : \mathbb{E} \to \mathbb{B}$ models a type theory with two sorts Kind and Type.

Category theory	Type theory
$\alpha \in \mathbb{B}$	α : Kind
$X \in \mathbb{E}_\alpha$	$X : \alpha \to$ Type

The Proposition says the pairs of $(\alpha : \text{Kind}, X : \alpha \to \text{Type})$ form a new type theory.

Proof of the Proposition, syntactically

Concepts	Definition
type	$(\alpha : \text{Kind}, X : \alpha \; ! \; \text{Type})$
element	$(a : \alpha, x : X(a))$
family	$(\beta : \alpha \; ! \; \text{Kind},$ $Y : \Pi_{a:\alpha}\beta(a) \; ! \; X(a) \; ! \; \text{Type})$
section	$(u : \Pi_{a:\alpha}\beta(a),$ $f : \Pi_{a:\alpha}\Pi_{x:X(a)}Y(a, u(a), x))$
pair	$((a, b) : \Sigma_{a:\alpha}\beta(a),$ $(x, y) : \Sigma_{x:X(a)}Y(a, b, x))$
identity type	?

Proof of the Proposition, syntactically

We need the path induction on an identity kind w.r.t. any type family over the identity kind.

$$
\frac{\begin{array}{l} \alpha : \text{Kind} \\ X : \Pi_{a,a':\alpha} a = a' \to \text{Type} \\ x : \Pi_{a:\alpha} X(a, a, \text{refl}_a) \end{array}}{\begin{array}{l} \text{ind}_= (X, x) : \Pi_{a,a':\alpha} \Pi_{p:a=a'} X(a, a', p) \\ \text{ind}_= (X, x, a, a, \text{refl}_a) \quad x \end{array}}
$$

In particular, for α : Kind, $X : \alpha \to$ Type and $p : a =_\alpha a'$, we have the *transport along* p

$p : X(a) \to X(a')$.

Proof of the Proposition, syntactically

The identity type of $(\alpha : \mathrm{Kind}, X : \alpha \to \mathrm{Type})$ is the pair of

- $=: \alpha \to \alpha \to \mathrm{Kind}$ and
- $\lambda a a' p x x'. p\ x = x' : \Pi_{a,a':\alpha} \Pi_{p:a=a'} X(a) \to X(a') \to \mathrm{Type}$.

It is the type of “path over path” but in different sorts.

Universes in a fibred setting

Let U : Kind and V : Type be universes of kinds and types. Then $(U, \lambda(\alpha : U).\alpha \ ! \ V)$ is a universe in the new type theory if, for any $\alpha : U$ and $X : \alpha \ ! \ V$, $\Pi_{a:\alpha}.X(a) : V$. Its elements are $(\alpha : U, X : \alpha \ ! \ V)$.

Equivalences in a fibred setting

For $(u : \alpha \to \beta, f : \Pi_{a:\alpha} X(a) \to Y(u(a)))$, An element of is-equiv$(u, f)$ is (v : homotopy inverse of u, g : homotopy inverse of f above v).

Lemma

Suppose the function extensionality holds. Then

$$\text{is-equiv}(u, f) \simeq (\text{is-equiv}(u), \lambda\_.\Pi_{a:\alpha}\text{is-equiv}(f_a))$$

for all $(u : \alpha \to \beta, f : \Pi_{a:\alpha} X(a) \to Y(u(a)))$ in the new type theory.

Univalence in a fibred setting

A universe U in a type theory is *univalent* if the canonical map $\lambda(A : U).(A, A, \mathrm{id}_A) : U \to \Sigma_{A,A':U} A \simeq A'$ is an equivalence.

The new universe $(U, \lambda(\alpha : U).\alpha \to V)$ is univalent if $\lambda(\alpha : U).(\alpha, \alpha, \mathrm{id}_\alpha)$ is an equivalence and for all $\alpha : U$, $\lambda X.(X, X, \lambda(a : \alpha).\mathrm{id}_{X(a)}) : (\alpha \to V) \to \Sigma_{X,Y:\alpha \to V} \Pi_{a:\alpha} X(a) \simeq Y(a)$ is an equivalence. This holds if U and V are univalent and the function extensionality holds.

Outline

Introduction

Foundations

Examples

Appendix

Arrow categories

Let $\mathbb{C}$ be a type-theoretic fibration category and write $(\mathbb{C}^{\to})_{\mathsf{f}}$ $\mathbb{C}^{\to}$ for the full subcategory of all the fibrations. Then **cod** : $(\mathbb{C}^{\to})_{\mathsf{f}} \to \mathbb{C}$ is a fibred type-theoretic fibration category. In this case Type Kind.

If $\mathbb{C}$ has a univalent universe, so does $(\mathbb{C}^{\to})_f$.

Originally, Shulman proved $(\mathbb{C}^{\to})_f$ is a type-theoretic fibration category [Shulman, 2015], and I give a fibred categorical description.

Change of base

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibred type-theoretic fibration category and $F : \mathbb{A} \to \mathbb{B}$ be a functor preserving fibrations, pullbacks of fibrations and acyclic cofibrations. Then the change of base or pullback

$$\begin{tikzcd} F\,\mathbb{E} \arrow[r] \arrow[d] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{A} \arrow[r, "F"'] & \mathbb{B} \end{tikzcd}$$

is a fibred type-theoretic fibration category.

Relational model

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibred type-theoretic fibration category. We have the change of base

$$\begin{tikzcd} Rel(p) \arrow[r] \arrow[d] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{B} \arrow[r, "\lambda\alpha.\alpha \quad \alpha"'] & \mathbb{B}. \end{tikzcd}$$

$Rel(p)$ is the category of binary families $(\alpha : \text{Kind}, R : \alpha \to \alpha \to \text{Type})$.

Application

Let $t : \Pi_{A:U}\Pi_{x:A} x = x \to x = x$.

Theorem

- t is “natural”: for any $f : A \to B$ in U, $\mathrm{ap}_f \quad t \quad t \quad \mathrm{ap}_f$.

$$
\begin{tikzcd}
x = x \arrow[r, "t"] \arrow[d, "\mathrm{ap}_f"'] & x = x \arrow[d, "\mathrm{ap}_f"] \\
fx = fx \arrow[r, "t"'] & fx = fx
\end{tikzcd}
$$

- If the type theory has $\mathbb{S}^1$, then for some $n \in \mathbb{Z}$, $tp = p^n$ for all $p : x = x$.

Application

Let $\mathbb{C}$ be the syntactic category of Martin-Löf type theory with a univalent universe U. Then $(U, \lambda(A, B : U).A \to B \to U)$ is a univalent universe in $Rel(\mathbf{cod})$. We have a strong fibration functor

$$\begin{tikzcd}
 & Rel(\mathbf{cod}) \arrow[d] \\
\mathbb{C} \arrow[ur, dotted, "R"] \arrow[r, equal] & \mathbb{C}.
\end{tikzcd}$$

In particular, for every closed term $t : A$, we have $R_A : A \to A \to \text{Type}$ and $R_t : R_A(t, t)$.

Application

Let $t : \Pi_{A:U}\Pi_{x:A}x = x \;\; ! \;\; x = x$. Then $R_t :$
$\Pi_{A,B:U,W:A \; ! \; B \; ! \; U}\Pi_{x:A,y:B,v:W(x,y)}\Pi_{p:x=x,q:y=y}(p, q) \;\; v =$
$v \;\; ! \;\; (tp, tq) \;\; v = v$.

$$
\begin{tikzcd}
x \arrow[r, equal, "p"] \arrow[d, dash, "v"'] \arrow[dr, phantom, "="] & x \arrow[d, dash, "v"] \\
y \arrow[r, equal, "q"'] & y
\end{tikzcd}
\quad ! \quad
\begin{tikzcd}
x \arrow[r, equal, "tp"] \arrow[d, dash, "v"'] \arrow[dr, phantom, "="] & x \arrow[d, dash, "v"] \\
y \arrow[r, equal, "tq"'] & y
\end{tikzcd}
$$

Application

Given $f : A \;\; ! \;\; B$ in U, let $W(x, y) \quad\;\; fx = y$. Then R_t looks like

$$
\begin{tikzcd}
fx \arrow[r, equal, "\mathsf{ap}_f\, p"] \arrow[d, equal, "v"'] \arrow[dr, phantom, "="] & fx \arrow[d, equal, "v"] \\
y \arrow[r, equal, "q"'] & y
\end{tikzcd}
\quad ! \quad
\begin{tikzcd}
fx \arrow[r, equal, "\mathsf{ap}_f(tp)"] \arrow[d, equal, "v"'] \arrow[dr, phantom, "="] & fx \arrow[d, equal, "v"] \\
y \arrow[r, equal, "tq"'] & y.
\end{tikzcd}
$$

Let $y \quad\;\; fx$, $q \quad\;\; \mathsf{ap}_f\, p$ and $v \quad\;\; \mathsf{refl}_{fx}$ and apply the function to $\mathsf{refl}_{\mathsf{ap}_f\, p}$. Then $\mathsf{ap}_f(tp) = t(\mathsf{ap}_f\, p)$ for all $x : A$ and $p : x = x$.

Application

Suppose the type theory has $\mathbb{S}^1 : U$ with $b : \mathbb{S}^1$ and $l : b = b$. Then $(\mathbb{S}^1, =_{\mathbb{S}^1})$ is a unit circle in $Rel(\mathbf{cod})$ and we still have the functor $R : \mathbb{C} \ !\ Rel(\mathbf{cod})$. Let $y : B$ and $q : y = y$ which corresponds to $f : \mathbb{S}^1 \ !\ B$. $t(l) = l^n$ for some $n \ 2 \ \mathbb{Z}$ and

$$
\begin{aligned}
tq &= t(\mathrm{ap}_f(l)) \\
&= \mathrm{ap}_f(t(l)) \\
&= \mathrm{ap}_f(l^n) \\
&= (\mathrm{ap}_f(l))^n \\
&= q^n
\end{aligned}
$$

References I

Hermida, C. (1993).
Fibrations, Logical Predicates and Indeterminates.
PhD thesis, University of Edinburgh.

Shulman, M. (2015).
Univalence for inverse diagrams and homotopy canonicity.
Mathematical Structures in Computer Science, 25(05):1203–1277.